Invariant Percolation and Harmonic Dirichlet Functions

Damien Gaboriau*

Abstract

The main goal of this paper is to answer question 1.10 and settle conjecture 1.11 of Benjamini-Lyons-Schramm [BLS99] relating harmonic Dirichlet functions on a graph to those on the infinite clusters in the uniqueness phase of Bernoulli percolation. We extend the result to more general invariant percolations, including the Random-Cluster model. We prove the existence of the nonuniqueness phase for the Bernoulli percolation (and make some progress for Random-Cluster model) on unimodular transitive locally finite graphs admitting nonconstant harmonic Dirichlet functions. This is done by using the device of ℓ^2 Betti numbers.

Mathematical Subject Classification: 60K35, 82B43, 31C05, 37A20, 05C25, 05C80, 37R30
Key words and phrases: percolation; transitive graph; harmonic Dirichlet function; measured equivalence relation; L^2 Betti number.

0 Introduction

Traditionally, percolation on graphs lives on $\mathbb{Z}^d$ or lattices in $\mathbb{R}^d$. Following earlier work of G. Grimmett and C. Newman [GN90] on the direct product of a regular tree and $\mathbb{Z}$, a general study of invariant percolation was initiated in I. Benjamini and O. Schramm [BS96] and further developed by several authors.

Let $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ be a (non-oriented) countable infinite locally finite graph. A **bond percolation** on $\mathcal{G}$ is simply a probability measure $\mathbf{P}$ on $\Omega = \{0,1\}^{\mathtt{E}}$, the subsets of its edge set $\mathtt{E}$. It is an **invariant percolation** when this measure is invariant under a certain group H of automorphisms of $\mathcal{G}$.

An element ω in Ω defines the **graph** whose vertices are $\mathtt{V}$ and whose edges are the **retained** (or **open**) edges, i.e. those $e \in \mathtt{E}$ with value $\omega(e) = 1$. It is the subgraph of $\mathcal{G}$ where edges with value 0 are **removed** (or **closed**). One is interested in the shape of the "typical" random subgraph ω [1] and of its **clusters**, i.e. its connected components.

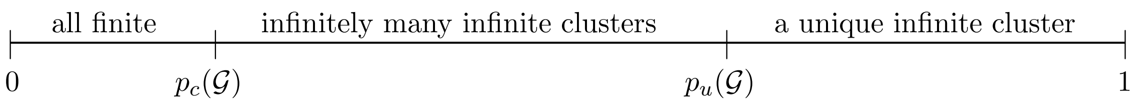
One of the most striking instances is **Bernoulli bond percolation**, and particularly on a **Cayley graph** [2] of a finitely generated group: each edge of $\mathcal{G}$ is removed with probability $1-p$ independently (where $p \in [0,1]$ is a parameter). The resulting probability measure μ_p on Ω is the product Bernoulli measure $(1-p, p)$ on $\{0,1\}$. It is invariant under every automorphism group of $\mathcal{G}$. How does the behavior evolve as p varies? For small p, the clusters are a.s. all finite, while for $p = 1$ the measure

*C.N.R.S.

[1] In more probabilistic terms, ω is a random variable with values in Ω and distribution $\mathbf{P}$.

[2] A Cayley graph will always be assumed to be for a finitely generated group and with respect to a *finite* generating system.

concentrates on the infinite subgraph $\mathcal{G}$ itself. Depending on the value of the parameter, μ_p-almost every subgraph $\omega \in \Omega$ has no infinite cluster, infinitely many infinite clusters (**nonuniqueness phase**) or only one infinite cluster (**uniqueness phase**). According to a somewhat surprising result of O. Häggström and Y. Peres [HP99], the phases are organized around two phase transitions for two critical values of p depending on the graph $0 < p_c(\mathcal{G}) \le p_u(\mathcal{G}) \le 1$, as summarized [3] in the following picture:



The picture at the critical values themselves is far from complete (to which interval belong the transitions? which inequalities are strict: $p_c \neq p_u \neq 1$?) and seems to depend heavily (for Cayley graphs) on the algebraic properties of the group. However, a certain amount of results has been obtained. For instance, in the Cayley graphs setting [4]:
• $p_u = p_c$ for amenable groups (Burton-Keane [BK89])
• $p_c < 1$ for groups of polynomial or exponential growth, except for groups with two ends [Lyo95, LP05]
• For any nonamenable group, there is almost surely no infinite cluster at $p = p_c$ [BLPS99a, Th. 1.3] [5]
• $p_u < 1$ for finitely presented groups with one end (Babson-Benjamini [BB99]) and for (restricted) wreath products [6] $K \wr \Lambda := \Lambda \ltimes \oplus_W K$ with finite non-trivial K (Lyons-Schramm [LS99])
• $p_u = 1$ for groups with infinitely many ends, thus the percolation at $p = p_u$ belongs to the uniqueness phase [7]
• The percolation at the threshold $p = p_u$ belongs to the nonuniqueness phase, and thus $p_u < 1$, for infinite groups with Kazhdan's property (T) (Lyons-Schramm [LS99])
• in the nonuniqueness phase, infinite clusters have uncountably many ends almost surely [HP99] [8]

For (much !) more information and references, the reader is referred to the excellent survey of R. Lyons [Lyo00], book (in preparation) by R. Lyons and Y. Peres [LP05] and papers [BLPS99a, BLPS99b, BLS99, BS96, HP99, LS99].

0.1 On Harmonic Dirichlet functions

The space $\mathbf{HD}(\mathcal{G})$ of **Harmonic Dirichlet functions** on a locally finite graph $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ is the space of functions on the vertex set $\mathtt{V}$ whose value at each vertex equals the average of the values at its neighbors

$$\left(\sum_{v' \sim v} 1\right) f(v) = \sum_{v' \sim v} f(v')$$

[3] For Cayley graphs, say. More generally this picture appears for Bernoulli percolation of unimodular quasi-transitive (see below for the definitions) graphs [HP99]. R. Schonmann has even removed the unimodularity assumption.

[4] i.e. for any Cayley graph $\mathcal{G}$ of a given finitely generated group Γ

[5] This is true more generally for nonamenable unimodular transitive graphs [BLPS99b].

[6] The finitely generated group Λ acts transitively on the discrete set W of indices and thus on the W-indexed direct product $\oplus_W K$.

[7] This is a very general result, see [LP05].

[8] This is true more generally for quasi-transitive unimodular graphs [HP99]. This has also been proved in the nonunimodular case by O. Häggström, Y. Peres, R. Schonmann [HPS99].

and whose coboundary is ℓ^2-bounded

$$\|df\|^2 = \sum_{v\sim v'} |f(v) - f(v')|^2 < \infty.$$

The constant functions on the vertex set $\mathtt{V}$ always belong to $\mathbf{HD}(\mathcal{G})$. Denote by $\mathcal{O}_{\mathbf{HD}}$ the class of connected graphs for which these are the only harmonic Dirichlet functions. Belonging or not to $\mathcal{O}_{\mathbf{HD}}$ plays a role in electrical networks theory: when assigning resistance 1 ohm to each edge, the coboundary of a harmonic Dirichlet function gives a finite energy current satisfying both Kirchhoff's laws.

As an example, a Cayley graph of a group Γ is in $\mathcal{O}_{\mathbf{HD}}$ if and only if the first ℓ^2 Betti number $\beta_1(\Gamma)$ of the group vanishes (see Theorem 6.1). Thus, the Cayley graphs of the following groups (when finitely generated) all belong to $\mathcal{O}_{\mathbf{HD}}$: abelian groups, amenable groups, groups with Kazhdan property (T), lattices in $\mathrm{SO}(n,1)$ ($n \geq 3$) or in $\mathrm{SU}(n,1)$. On the other hand, the class of groups whose Cayley graphs don't belong to $\mathcal{O}_{\mathbf{HD}}$ contains the non-cyclic free groups, the fundamental groups of surfaces of genus $g \geq 2$, the free products of infinite groups, and the amalgamated free products over an amenable group of groups in that class. Look at the very informative paper by Bekka-Valette [BV97] and F. Martin's thesis [Mar03] for further interpretations in cohomological terms.

P. Soardi [Soa93] has proved that belonging to $\mathcal{O}_{\mathbf{HD}}$ is invariant under a certain kind of "perturbation" of $\mathcal{G}$, namely quasi-isometry or rough isometry. Bernoulli bond percolation clusters may also be considered as perturbations of $\mathcal{G}$. I. Benjamini, R. Lyons and O. Schramm addressed the analogous invariance problem [9] in [BLS99], by taking a stand only in one case:

> **Question [BLS99, Quest. 1.10]** Let $\mathcal{G}$ be a Cayley graph, and suppose that $\mathcal{G} \in \mathcal{O}_{\mathbf{HD}}$. Let ω be Bernoulli percolation on $\mathcal{G}$ in the uniqueness phase. Does it follow that a.s. the infinite cluster of ω is in $\mathcal{O}_{\mathbf{HD}}$?
>
> **Conjecture [BLS99, Conj. 1.11]** Let $\mathcal{G}$ be a Cayley graph, $\mathcal{G} \notin \mathcal{O}_{\mathbf{HD}}$. Then a.s. all infinite clusters of p-Bernoulli percolation are not in $\mathcal{O}_{\mathbf{HD}}$.

They proved this for p sufficiently large:

> **Theorem[BLS99, Th. 1.12]** *If a Cayley graph $\mathcal{G}$ is not in $\mathcal{O}_{\mathbf{HD}}$, then there exists a $p_0 < 1$, such that every infinite cluster of μ_p-a.e. subgraph is not in $\mathcal{O}_{\mathbf{HD}}$, for every $p \geq p_0$.*

On the other hand,

> **Theorem[BLS99, Th. 1.9]** *If $\mathcal{G}$ is a Cayley graph of an **amenable** [10] group, then every cluster of μ_p-a.e. subgraph is in $\mathcal{O}_{\mathbf{HD}}$.*

The main goal of this paper is to complete these results and prove the following:

Theorem 0.1 *(Theorem 1.5) Let $\mathcal{G}$ be a Cayley graph of a finitely generated group. Consider Bernoulli percolation in the uniqueness phase. Then μ_p a.s. the infinite cluster ω_∞ of ω satisfies:*
ω_∞ has no harmonic Dirichlet functions besides the constants if and only if $\mathcal{G}$ has no harmonic Dirichlet functions besides the constants: μ_p a.s.

$$\omega_\infty \in \mathcal{O}_{\mathbf{HD}} \Longleftrightarrow \mathcal{G} \in \mathcal{O}_{\mathbf{HD}}$$

[9] Observe that for $p_u < p < 1$, the infinite clusters are μ_p a.s. not quasi-isometric to $\mathcal{G}$. They contain for instance arbitrarily long arcs without branch points (by deletion tolerance !).

[10] thus $\mathcal{G}$ is in $\mathcal{O}_{\mathbf{HD}}$

This answers question 1.10 and settles conjecture 1.11 of [BLS99]. Together with their Corollary 4.7 about the nonuniqueness phase, this theorem allows to complete the picture for Cayley graphs:

Corollary 0.2 *Let $\mathcal{G}$ a Cayley graph of a finitely generated group.*

- *If $\mathcal{G}$ is not in $\mathcal{O}_{\mathbf{HD}}$ (i.e. $\beta_1(\Gamma) \neq 0$), then a.s.*[11] *the infinite clusters of Bernoulli bond percolation (for both nonuniqueness and uniqueness phases) are not in $\mathcal{O}_{\mathbf{HD}}$.*
- *If $\mathcal{G}$ is in $\mathcal{O}_{\mathbf{HD}}$ (i.e. $\beta_1(\Gamma) = 0$), then a.s.*[11] *the infinite clusters of Bernoulli bond percolation are*
  – *not in $\mathcal{O}_{\mathbf{HD}}$ in the nonuniqueness phase*
  – *in $\mathcal{O}_{\mathbf{HD}}$ in the uniqueness phase.*

Since percolation at p_c belongs to the finite phase for nonamenable Cayley graphs, it turns out that every $p_0 > p_c$ suits in Th. 1.12 of [BLS99] recalled above, while $p_0 = p_c$ doesn't.

In the course of the proof, a crucial use is made of the notion of (first) L^2 Betti numbers for measured equivalence relations, introduced in [Gab02]. We consider two standard equivalence relations with countable classes associated with our situation: the **full equivalence relation** $\mathcal{R}^{\mathrm{fu}}$ and the **cluster equivalence relation** $\mathcal{R}^{\mathrm{cl}}$. They are defined concretely or also more geometrically (see Section 1.1 and 1.2) in terms of two *laminated spaces* $\mathcal{L}^{\mathrm{fu}}$ and $\mathcal{L}^{\mathrm{cl}}$, constructed from $\Omega \times \mathcal{G}$ after taking the quotient under the diagonal Γ-action and removing certain edges. Their laminated structure comes from the fact that these spaces are equipped with a measurable partition into *leaves*, corresponding to the decomposition of $\Omega \times \mathcal{G}$ into the graphs $\{\omega\} \times \mathcal{G} \simeq \mathcal{G}$.

One shows (Section 7) that the first L^2 Betti number of such an equivalence relation, generated by such a 1-dimensional lamination (in fact generated by a **graphing** in the sense of [Lev95, Gab00] or Section 8 or example 7.1 below), vanishes if and only if the leaf of almost every point in the transversal is a graph without harmonic Dirichlet functions, besides the constants:

Theorem 0.3 *(Corollary 7.6) Let $\mathcal{R}$ be a measure-preserving equivalence relation on the standard Borel probability measure space (X, μ). Let Φ be a graphing generating $\mathcal{R}$. If the graph $\Phi[x]$ associated with $x \in X$ has μ a.s. bounded degree*[12]*, then $\beta_1(\mathcal{R}, \mu) = 0$ if and only if μ a.s. $\mathbf{HD}(\Phi[x]) = \mathbb{C}$.*

Now the triviality of the first L^2 Betti number of an equivalence relation is invariant when taking a restriction to a Borel subset that meets almost every equivalence class [Gab02, Cor. 5.5]. Denote by U the subset of ω's with a **unique** infinite cluster and such that the base point ρ belongs to that cluster. In the uniqueness phase, U meets almost every $\mathcal{R}^{\mathrm{fu}}$-equivalence class and the restrictions of $\mathcal{R}^{\mathrm{fu}}$ and $\mathcal{R}^{\mathrm{cl}}$ to U define the same equivalence relation. And Theorem 0.1 follows.

Invariant bond percolation on a locally finite graph $\mathcal{G}$, for a group H of automorphisms of $\mathcal{G}$, is also considered in a more general setting than just Cayley graphs. For the invariance property of the measure to be of any use, the group has to be big enough. The standard hypothesis is that H is **transitive** or at least **quasi-transitive** (there is only one, resp. only finitely many orbits of vertices).

When closed in all automorphisms of $\mathcal{G}$, the group H is locally compact and equipped with a unique (up to multiplication by a constant) left invariant Haar measure. If that measure is also right invariant, then H is called **unimodular**. A graph with a unimodular quasi-transitive group H, is

[11] every infinite cluster for μ_p-almost every subgraph $\omega \in \{0,1\}^{\mathrm{E}}$

[12] a bound on the number of neighbors of each vertex

called itself unimodular. The unimodularity assumption is a quite common hypothesis in invariant percolation theory, where it is used in order to apply a simple form of the *mass-transport principle* (see for example [BLPS99a, sect. 3] and Section 2.3).

The same unimodularity assumption appears here, for a related reason: in order to ensure that a certain equivalence relation preserves the measure (see Section 2.3). We obtain the following generalization [13] of Theorem 0.1:

Theorem 0.4 *Let $\mathcal{G}$ be a locally finite graph, H a closed transitive unimodular group of automorphisms of $\mathcal{G}$ and* $\mathbf{P}$ *any H-invariant percolation. On the Borel subset of subgraphs with finitely many infinite clusters,* $\mathbf{P}$*-almost surely the infinite clusters belong (resp. do not belong) to $\mathcal{O}_{\mathbf{HD}}$ if and only if $\mathcal{G}$ belongs (resp. doesn't belong) to $\mathcal{O}_{\mathbf{HD}}$.*

The clusters of this theorem satisfy the more general property (to be introduced in Section 3.2) of being *(virtually) selectable* and the proof is given in that context (Section 3, Theorem 3.9).

The most studied invariant percolation, beyond Bernoulli, is probably the **Random-Cluster Model**. It was introduced by C. Fortuin and P. Kasteleyn [FK72] in relation with Ising and Potts models as explained for instance in [HJL02a, Prop. 2.3 and 2.4].

It is a (non-independent) percolation process, governed by two parameters [14] $p \in [0,1]$ and $q \in [1,\infty]$. It is defined through a limit procedure by considering an exhaustion $\mathcal{G}_m$ of $\mathcal{G}$ by finite subgraphs, and on the set of subgraphs of $\mathcal{G}_m$, this measure only differs from the Bernoulli(p) product measure by the introduction of a weight (q to the power the number of clusters). However, the count of this number of clusters is influenced by the boundary conditions. This leads to two particular incarnations of the Random-Cluster model: $\mathtt{WRC}_{p,q}$ and $\mathtt{FRC}_{p,q}$ according to the *Wired* (the boundary points are all connected from the exterior) or *Free* (there is no outside connection between the boundary points) boundary conditions . These invariant bond percolations both exhibit phase transitions, for each q, similar to that of Bernoulli percolation, leading to critical values $p_c(q)$ and $p_u(q)$ (denoted more precisely by $p_c^{\mathtt{W}}(q)$, $p_u^{\mathtt{W}}(q)$ and $p_c^{\mathtt{F}}(q)$, $p_u^{\mathtt{F}}(q)$ in case the boundary conditions have to be emphasized). They "degenerate" to Bernoulli percolation when $q = 1$.

The reader is invited to consult the papers [HJL02a, HJL02b] of O. Häggström, J. Jonasson and R. Lyons, for most of the results relevant for this paper and for details and further references.

The above Theorem 0.4 obviously specializes to:

Corollary 0.5 *Let $\mathcal{G}$ be a locally finite graph admitting a transitive unimodular group of automorphisms. Consider the Random-Cluster model $\mathtt{RC}_{p,q} = \mathtt{WRC}_{p,q}$ or $\mathtt{FRC}_{p,q}$ in the uniqueness phase. Then $\mathtt{RC}_{p,q}$ a.s. the infinite cluster admits non- (resp. only) constant harmonic Dirichlet functions if and only if $\mathcal{G}$ admits non- (resp. only) constant harmonic Dirichlet functions.*

[13] Compare with [BLS99] where Theorem 5.7 extends Theorem 1.12 (recalled above) to the more general setting of a unimodular transitive graph, and for much more general percolations than Bernoulli percolation.

[14] The temperature, T in Ising or Potts models is linked with the parameter p of the Random-Cluster model by $p = 1 - e^{-\frac{2}{T}}$. The parameter q, taken to be $q = 2$ in the Ising Model, resp. $q \in \mathbb{N}$ in the Potts Model, may assume any value in $[1,\infty)$ for the Random-Cluster model. For example, the (free) Gibbs distribution $\mathtt{FPt}_{\frac{1}{T},q}$ of the Potts model on $\{0,1\}^{\mathtt{V}}$ is obtained from $\mathtt{FRC}_{p,q}$ by choosing a subgraph $\omega \in \{0,1\}^{\mathtt{E}}$ according to $\mathtt{FRC}_{p,q}$ and then choosing a color in $\{1,2,\cdots,q\}$ uniformly and independently on the vertices of each cluster.

0.2 On the nonuniqueness Phase

One of the most famous conjectures in the subject is probably Conjecture 6 of Benjamini and Schramm [BS96]:

The nonuniqueness phase always exists[15] *for Cayley graphs $\mathcal{G}$ of nonamenable groups.*

and more generally

If the quasi-transitive graph $\mathcal{G}$ has positive Cheeger constant, then $p_c(\mathcal{G}) < p_u(\mathcal{G})$.

I. Pak and T. Smirnova-Nagnibeda [PSN00] proved that each finitely generated nonamenable group admits a Cayley graph for which $p_c < p_u$. On the other hand, the groups with *cost* strictly bigger than 1 (see [Gab00] or Section 8 below, item "cost") are the only ones for which it is known that $p_c \neq p_u$ for every Cayley graph (R. Lyons [Lyo00]). This class of Cayley graphs contains all those outside $\mathcal{O}_{\mathbf{HD}}$ (see Th. 6.1 and [Gab02, Cor. 3.23]), but it is unknown whether the reverse inclusion holds.

We are able, using our ℓ^2 methods, to extend Lyons' result to the unimodular setting and to make some progress for Random-Cluster model. Our treatment doesn't make use of the continuity of the probability that ρ belongs to an infinite cluster, but only of the expected degree. We show:

Theorem 0.6 *(Cor. 4.5) Let $\mathcal{G}$ be a unimodular transitive locally finite graph. If $\mathcal{G}$ doesn't belong to $\mathcal{O}_{\mathbf{HD}}$, then the nonuniqueness phase interval of Bernoulli percolation has non-empty interior:*

$$p_c(\mathcal{G}) < p_u(\mathcal{G})$$

In fact, to each unimodular transitive locally finite graph $\mathcal{G}$, we associate (see def. 2.10) a *numerical invariant* $\beta_1(\mathcal{G})$, which can be interpreted as the first ℓ^2 Betti number of any closed transitive group of automorphisms of $\mathcal{G}$. *It vanishes if and only if $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$*. In case $\mathcal{G}$ is a Cayley graph of a group Γ, then $\beta_1(\mathcal{G}) = \beta_1(\Gamma)$. A transitive tree of degree d has $\beta_1(\mathcal{G}) = \frac{d}{2} - 1$.

For Bernoulli percolation, we get the more precise estimate, where $\deg(\mathcal{G})$ denotes the degree of a (any) vertex of $\mathcal{G}$:

$$0 < \beta_1(\mathcal{G}) \leq \frac{1}{2}\deg(\mathcal{G})(p_u(\mathcal{G}) - p_c(\mathcal{G})).$$

Observe that $\deg(\mathcal{G})\, p$ is the expected degree $\mu_p[\deg(\rho)]$ of a base point ρ with respect to the Bernoulli measure of parameter p. These inequalities appear as Corollary 4.5 of a quite general result (Th. 4.2) which applies to more general percolations, like the free or the wired Random-Cluster model, $\mathtt{RC}_{p,q} = \mathtt{WRC}_{p,q}$ or $\mathtt{FRC}_{p,q}$:

Theorem 0.7 *(Cor. 4.7) Let $\mathcal{G}$ be a unimodular transitive locally finite graph, not in $\mathcal{O}_{\mathbf{HD}}$. Fix the parameter $q \in [1, \infty)$. The gap between the left limit (when $p \nearrow p_c(q)$) and the right limit (when $p \searrow p_u(q)$) of the expected degree of a base point ρ with respect to the measure $\mathtt{RC}_{p,q}$ satisfies:*

$$0 < \beta_1(\mathcal{G}) \leq \frac{1}{2}\Big(\mathtt{RC}_{p_u+,q}[\deg(\rho)] - \mathtt{RC}_{p_c-,q}[\deg(\rho)]\Big).$$

[15] i.e. $p_c(\mathcal{G}) < p_u(\mathcal{G})$ for Bernoulli percolation

Indeed the function $p \mapsto \mathtt{FRC}_{p,q}[\deg(\rho)] := \int_{\{0,1\}^{\mathtt{E}}} \deg(\rho)(\omega) d\mathtt{FRC}_{p,q}(\omega)$ is left continuous while $p \mapsto \mathtt{WRC}_{p,q}[\deg(\rho)]$ is right continuous in p, but the possible remaining discontinuity doesn't allow to conclude in general that $p_c(q) < p_u(q)$. Of course, $\mathtt{RC}_{p_u+,q}[\deg(\rho)]$ denotes $\lim_{p \searrow p_u(q)} \mathtt{RC}_{p,q}[\deg(\rho)]$ and $\mathtt{RC}_{p_c-,q}[\deg(\rho)] := \lim_{p \nearrow p_c(q)} \mathtt{RC}_{p,q}[\deg(\rho)]$.

There is no fundamental qualitative difference when shifting from transitive to quasi-transitive graphs. The slight modifications are presented in section 5.

0.3 About Higher Dimensional Invariants and Treeablility

Higher dimensional ℓ^2 Betti numbers are also relevant in percolation theory. Y. Peres and R. Pemantle [PP00] introduced the percolation theoretic notion of countable **treeable groups**. They are groups Γ for which the space of trees with vertex set Γ admits a Γ-invariant probability measure. They proved that nonamenable direct products are not treeable. It is not hard to show that being treeable is equivalent to being *not anti-treeable* in the sense of [Gab00, Déf. VI.1] or to having *ergodic dimension* 1 in the sense of [Gab02, Déf. 6.4].

Similarly, R. Lyons introduced the notion of **almost treeable groups**: They are groups for which the space of forests with vertex set Γ admits a sequence $(\mathbf{P}_n)_{n\in\mathbb{N}}$ of Γ-invariant probability measures with the property that for each pair of vertices, the probability that they belong to the same connected component of the forest tends to 1 as n tends to infinity: $\forall \gamma_1, \gamma_2 \in \Gamma$, $\lim_{n\to\infty} \mathbf{P}_n(\gamma_1 \leftrightarrow \gamma_2) = 1$. Clearly treeable implies almost treeable.

It is not hard to show that Γ is almost treeable if and only if it has *approximate ergodic dimension* 1; where the **approximate ergodic dimension** of Γ is the minimum of the approximate dimensions of the equivalence relations produced by a free p.m.p. action of Γ on a standard Borel space (see [Gab02, Déf. 5.15]).

The next theorem follows from [Gab02, Cor. 5.13, Prop. 5.16, Prop. 6.10] and imposes serious restrictions for a group to be treeable or almost treeable. In particular, lattices in $\mathrm{SO}(n,1)$ are treeable if and only if $n \leq 2$. Also direct products $\Gamma_1 \times \Gamma_2$ are not almost treeable as soon as Γ_1 and Γ_2 contain a copy of the free group $\mathbf{F}_2$. This answer questions of R. Lyons and Y. Peres (personal communication).

Theorem 0.8 *If Γ is treeable in the sense of [PP00], then $\beta_1(\Gamma) = 0$ if and only if Γ is amenable. If Γ is almost treeable, then its higher ℓ^2 Betti numbers all vanish: $\beta_n(\Lambda) = 0$ for every $n \geq 2$.*

Warning for the reader Theorem 0.1 is clearly a specialization of Theorem 0.4. However, for the convenience of the reader mainly interested in Cayley graphs and also to serve as a warm-up for the more technical general case, we present first a separate proof of Theorem 0.1 (Section 1, Subsection 1.3.b and Th. 1.5), while Theorem 0.4 is proved in Section 3.3. A necessary consequence is a certain number of repetitions. Section 4 is devoted to the proof of Theorems 0.7 and 0.6. Some notions related to equivalence relations are recalled in Section 8. Theorem 0.3 is used at several places. It is proved as Corollary 7.6 (see also Remark 7.7). But the sections 6 and 7 are quite technical, and I put it back until the end of the paper. It may be a good advice to skip them and to keep Theorem 7.5 and Corollary 7.6 as "black boxes" for a first reading.

Acknowledgment I wish to thank Thierry Barbot, Étienne Ghys, Alekos Kechris, Yuval Peres and Bruno Sévennec for many valuable discussions, and especially Vincent Beffara who helped me to

understand several references. I am particularly grateful to Russell Lyons who explained to me many entertaining and impressive results in graph percolation theory, as well as some connections with my previous work on cost of equivalence relations, and who suggested that the notion of ℓ^2 Betti numbers of equivalence relations could be relevant for question 1.10 and conjecture 1.11 of [BLS99]. Thanks also for his careful proofreading.

My acknowledgments are due to the anonymous referee who recommended the addition of a section about the quasi-transitive case (Section 5 of the current version) and for his accurate and pertinent comments.

To Carole, Valentine, Alice, Lola and Célestin.

Contents

1 Percolation on Cayley Graphs

Let $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ be a Cayley graph of a finitely generated group Γ. Let ρ be a base vertex, for example the vertex representing the identity element of Γ. The group Γ acts freely on the set $\mathtt{E}$ of edges and freely transitively on the set $\mathtt{V}$ of vertices.

The space $\Omega := \{0,1\}^{\mathtt{E}}$ is the space of **colorings** (assignment of a number) of the edges of $\mathcal{G}$ with two colors (0 and 1) of $\mathcal{G}$. A point $\omega \in \Omega$ is also the characteristic function of a subset of $\mathtt{E}$. When viewed as a subgraph of $\mathcal{G}$ it is denoted by $\omega(\mathcal{G})$. It then has the same set of vertices $\mathtt{V}$ as $\mathcal{G}$ and for edges the set of **retained** or **open** edges: those edges $e \in \mathtt{E}$ with color $\omega(e) = 1$. It has the same base vertex ρ as $\mathcal{G}$. The **cluster** $\omega(v)$ of a vertex v is its connected component in $\omega(\mathcal{G})$. The action of Γ on $\mathtt{E}$ induces an action [16] on the space Ω of colorings.

Let (X, μ) be a standard Borel probability space together with

- a probability measure-preserving (p.m.p.) action of Γ, which is *(essentially) free* [17], and
- a Γ-equivariant Borel map $\pi : X \to \{0,1\}^{\mathtt{E}}$.

The push-forward measure $\pi_*\mu$ is a **Γ-invariant bond percolation** on $\mathcal{G}$.

1.1 The Full Equivalence Relation

Consider now the space $X \times \mathcal{G}$ with the diagonal action of Γ. It is a "laminated space", with leaves $\{x\} \times \mathcal{G}$.

Dividing out by the diagonal action of Γ, one gets the laminated space $\mathcal{L}^{\mathrm{fu}} = \Gamma\backslash(X \times \mathcal{G})$: the **full lamination**. It is a (huge, highly disconnected) graph with vertex set $\Gamma\backslash(X \times \mathtt{V})$ and edge set $\Gamma\backslash(X \times \mathtt{E})$. A **leaf** is a connected component of this graph.
– Because of the freeness of the Γ-action on $\mathtt{V}$, the image $X^\bullet$ in $\mathcal{L}^{\mathrm{fu}}$ of the space $X \times \{\rho\}$ is an embedding, leading to a natural identification of X with $X^\bullet$.
– Because of the transitivity on $\mathtt{V}$ of the Γ-action, $X^\bullet$ equals $\Gamma\backslash(X \times \mathtt{V})$.

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X^\bullet = \Gamma\backslash(X \times \mathtt{V}) \\ x & \longmapsto & (x, \rho) \sim (\gamma x, \gamma\rho) \end{array}$$

Let's denote by $\mu^\bullet$ the push-forward of the measure μ to $X^\bullet$.

With the (any) choice of ρ, the Γ-set $\mathtt{V}$ identifies with Γ equipped with the action by left multiplication and the left action of Γ on itself, by multiplication by the inverse on the right, induces on

[16] For $\gamma \in \Gamma$: $\omega' = \gamma \cdot \omega$ if and only if $\omega'(e) = \omega(\gamma^{-1}e)$ for every edge $e \in \mathtt{E}$.
[17] the Borel set of points $x \in X$ with non-trivial stabilizer have μ-measure 0

$X^\bullet = \Gamma\backslash(X \times \mathtt{V}) \simeq \Gamma\backslash(X \times \Gamma)$ a $\mu^\bullet$-preserving Γ-action, isomorphic to the original one on X. With $\rho \leftrightarrow 1$ and $\gamma\rho \leftrightarrow \gamma$ one gets $\gamma_1(x, 1) := (x, 1\gamma_1^{-1}) = (x, \gamma_1^{-1}1) \sim (\gamma_1(x), 1)$.
– Because of the freeness of the Γ-action on X, the leaf of $\mu^\bullet$-almost every $x^\bullet \in X^\bullet$ is isomorphic to $\mathcal{G}$.

Definition 1.1 *Define the* ***full equivalence relation*** $\mathcal{R}^{\mathrm{fu}}$ *on* $X^\bullet$ *by* $x^\bullet \mathcal{R}^{\mathrm{fu}} y^\bullet$ *if and only if* $x^\bullet$ *and* $y^\bullet$ *are vertices of the same* $\mathcal{L}^{\mathrm{fu}}$*-leaf.*

It is isomorphic with that given by the Γ-action on X and preserves the measure.

$\mathcal{R}^{\mathrm{fu}}$: *Two points* x, y *are* $\mathcal{R}^{\mathrm{fu}}$*-equivalent if and only if there is* $\gamma \in \Gamma$ *such that* $\gamma x = y$*.*

Example 1.2 *Let* $\mathcal{G} = L$ *be the standard Cayley graph of* $\mathbb{Z}$*: the simplicial line. Take* ρ *as the point* 0 *of the line. The product space* $X \times L$ *is a kind of cylinder laminated by lines. A fundamental domain for the diagonal* $\mathbb{Z}$*-action on* $X \times L$ *is given by* $X \times [0, 1)$*. Denote by* t *the automorphism of* X *given by the generator* 1 *of* $\mathbb{Z}$*. The quotient space* $\mathbb{Z}\backslash(X \times L)$ *identifies with the usual* suspension *or* mapping torus *construction* $(\omega, 0) \sim (t\omega, 1)\backslash(X \times [0, 1])$ *obtained from* $X \times [0, 1]$ *(laminated by* $\{\omega\} \times [0, 1]$*) by gluing together the top and bottom levels* $X \times \{0\}$ *and* $X \times \{1\}$ *after twisting by* t*. The gluing scar is the transversal* $X^\bullet$*.*

1.2 The Cluster Equivalence Relation

Now, thanks to the map $\pi : X \to \{0, 1\}^{\mathtt{E}}$, the field of graphs $x \mapsto \{x\} \times \mathcal{G}$ becomes a Γ-equivariant field of colored graphs $x \mapsto \pi(x)$. Each leaf of $\mathcal{L}^{\mathrm{fu}}$ becomes a colored graph.

By removing all the 0-colored edges, one defines a subspace $\mathcal{L}^{\mathrm{cl}}$ of $\mathcal{L}^{\mathrm{fu}}$: the **cluster lamination**. A leaf of $\mathcal{L}^{\mathrm{cl}}$ is a connected component of 1-colored (or retained) edges.

Definition 1.3 *Define the* ***cluster equivalence relation*** $\mathcal{R}^{\mathrm{cl}}$ *on* $X^\bullet$ *by* $x^\bullet \mathcal{R}^{\mathrm{cl}} y^\bullet$ *if and only if* $x^\bullet$ *and* $y^\bullet$ *are vertices of the same* $\mathcal{L}^{\mathrm{cl}}$*-leaf. It is a subrelation of* $\mathcal{R}^{\mathrm{fu}}$*.*

For μ-almost every $x \in X$, the leaf of $x^\bullet$ is isomorphic to the cluster $\mathcal{G}_x := \pi(x)(\rho)$ of the vertex ρ in the subgraph $\pi(x)$ of $\mathcal{G}$. Thus the $\mathcal{R}^{\mathrm{cl}}$-class of $x^\bullet$ is infinite if and only if the corresponding cluster $\pi(x)(\rho)$ is infinite. For each $x^\bullet \in X^\bullet$, the family of $\mathcal{R}^{\mathrm{cl}}$-classes into which its $\mathcal{R}^{\mathrm{fu}}$-class decomposes is in natural bijection with the clusters of $\pi(x)$. The $\mathcal{R}^{\mathrm{fu}}$-class of $x^\bullet$ contains n infinite $\mathcal{R}^{\mathrm{cl}}$-classes iff $\pi(x)$ has n infinite clusters.

Let $e = [\rho, \gamma^{-1}\rho]$ be an edge with end point ρ. Once descended in $\mathcal{L}^{\mathrm{fu}}$, the edge $\{x\} \times e = [(x, \rho), \underbrace{(x, \gamma^{-1}\rho)}_{\sim(\gamma x, \rho)}]$ is retained in $\mathcal{L}^{\mathrm{cl}}$ iff $\pi(x)(e) = 1$. In this case, the vertices $\rho, \gamma^{-1}\rho$ are in the same cluster of $\pi(x)$, while $x, \gamma x$ are $\mathcal{R}^{\mathrm{cl}}$-equivalent. More generally:

$\mathcal{R}^{\mathrm{cl}}$: *Two points* x, y *are* $\mathcal{R}^{\mathrm{cl}}$*-equivalent if and only if there is* $\gamma \in \Gamma$ *such that* $\gamma x = y$ *and the vertices* $\rho, \gamma^{-1}\rho$ *are in the same cluster of* $\pi(x)$*.*

It may be relevant to emphasize the role of π and include it in the notation: $\mathcal{R}^{\mathrm{cl}}_\pi$.

Example 1.2 (continued) *The edges are divided into the positive ones and the negative one according to their position with respect to* $\rho = 0$*. The* $\mathcal{R}^{\mathrm{cl}}$*-equivalence class of* $x \in X$ *consists of the iterates* $t^k(x)$ *for* $k \in \{-i, -i+1, \cdots, 0, 1, 2, \cdots, j-1, j\}$*, where* j *and* i *are the number of negative and positive*

edges colored 1 *in* $\pi(x)$*, starting from* ρ*:*

$$\pi(x) = (\overbrace{\cdots,0,\underbrace{1,1,1,\cdots,1}_{j \text{ times}}}^{\text{negative edges}},\overbrace{\underbrace{1,1,1,\cdots,1}_{i \text{ times}},0,\cdots}^{\text{positive edges}}).$$

From now on, and until the end of Section 1 we won't distinguish between (X,μ) *and* $(X^\bullet,\mu^\bullet)$.

The **uniqueness set** U is the Borel subset of points ω of Ω such that ω has a **unique** infinite cluster and such that ρ belongs to it. The **uniqueness set** U^π **of** π is the Borel subset of points of X such that $\pi(x)$ belongs to U.

Proposition 1.4 *When restricted to the uniqueness set* U^π *the two equivalence relations* $\mathcal{R}^{\mathrm{fu}}$ *and* $\mathcal{R}^{\mathrm{cl}}$ *do coincide.*

The point here is the *selectability* of the infinite cluster (see Subsection 3.2 devoted to that subject).

PROOF: Let x,y be two $\mathcal{R}^{\mathrm{fu}}$-equivalent points of U^π. Since $\pi(x)$ contains only one infinite cluster, the $\mathcal{R}^{\mathrm{fu}}$-class of x contains only one infinite $\mathcal{R}^{\mathrm{cl}}$-class. The $\mathcal{R}^{\mathrm{cl}}$-equivalence classes of x and y being both infinite, they coincide. ■

1.3 Examples

1.3.a Trivial Example

Apply the above construction to the particular constant map $x \overset{\pi_1}{\mapsto} \mathcal{G}$, sending every point x to the full graph $\mathcal{G}$ ($\omega \equiv 1$, i.e. $\pi(x)(e) = 1$ for every $x \in X$ and $e \in \mathtt{E}$, which is fixed by the whole of Γ). In this case, almost every $\mathcal{G}_x$ is just $\mathcal{G}$, the uniqueness set U^π equals X and $\mathcal{R}^{\mathrm{cl}} = \mathcal{R}^{\mathrm{fu}}$.

1.3.b Bernoulli Percolation

The main example is given by $X = \{0,1\}^{\mathtt{E}}$ itself, $\pi = id$ and $\mu = \mu_p$ the Bernoulli measure with survival parameter p, i.e. μ_p is the product of the measures giving weights $1-p, p$ to $0,1$. The Γ-action is essentially free for $p \neq 0$ or 1.

Stricto sensu, this example is just what is needed for the statement of Theorem 0.1. However, it is useful to introduce more objects in order to better distinguish between the various roles played by the space Ω.

The parameter p **belongs to the uniqueness phase** if and only if μ_p-almost every graph in Ω has a unique infinite cluster. In this phase, the uniqueness set U has non-zero μ_p-measure and is μ_p-a.s. the union of the infinite $\mathcal{R}^{\mathrm{cl}}$-classes. The restrictions of $\mathcal{R}^{\mathrm{fu}}$ and $\mathcal{R}^{\mathrm{cl}}$ to U coincide (Proposition 1.4). Theorem 0.1 of the introduction will thus be a corollary of Theorem 1.5 for $p < 1$ and is trivial for $p = 1$.

1.3.c Actions Made Free

If one is considering a percolation for which the Γ-action is not free, one can switch to a free action by taking any probability measure-preserving free Γ-action on a space (Y,ν), and replacing Ω by its product with Y, equipped with the product measure and the measure-preserving diagonal action of Γ, together with the natural Γ-equivariant projection $\pi : X = \Omega \times Y \to \Omega$. General (non-free) percolations are thus treated together in the same framework.

1.3.d Standard Coupling

The standard coupling is a very useful way to put all the Bernoulli measures μ_p together and to vary the map π instead of changing the measure on Ω with the parameter p.

Let $X = [0,1]^{\mathtt{E}}$ be the product space with the product measure μ of Lebesgue measures on the intervals $[0,1]$. An element of X gives a **colored graph**: a coloring of the graph $\mathcal{G}$, with $[0,1]$ as set of colors. For each $p \in [0,1]$, let π_p be the Γ-equivariant map sending $[0,1]$-colored graphs to $\{0,1\}$-colored ones by retaining only the edges colored in $[0,p]$:

$$\pi_p : \left(\begin{array}{ccc} [0,1]^{\mathtt{E}} & \longrightarrow & \{0,1\}^{\mathtt{E}} \\ x & \longmapsto & \pi_p(x) : \begin{cases} \pi_p(x)(e) = 1 & \text{if } x(e) \in [0,p] \\ \pi_p(x)(e) = 0 & \text{if } x(e) \in (p,1] \end{cases} \end{array} \right)$$

Clearly, π_p pushes the measure μ to μ_p. For each value of p, one gets the cluster equivalence relation $\mathcal{R}^{\mathrm{cl}}_p$, also defined as follows:

$\mathcal{R}^{\mathrm{cl}}_p$: *Two $[0,1]$-colored graphs x, y are $\mathcal{R}^{\mathrm{cl}}_p$-equivalent if and only if there is $\gamma \in \Gamma$ such that $\gamma x = y$ and the vertices $\rho, \gamma^{-1}\rho$ are connected in the colored graph x by a path of edges with colors $\leq p$.*

This gives an intuitive picture of the clusters evolution as p varies: The family $(\mathcal{R}^{\mathrm{cl}}_p)_{p\in[0,1]}$ is strictly increasing. Moreover for every p, $\mathcal{R}^{\mathrm{cl}}_p = \cup_{t<p}\mathcal{R}^{\mathrm{cl}}_t$ and $\mathcal{R}^{\mathrm{cl}}_1 = \mathcal{R}^{\mathrm{fu}}$. The critical value p_c is characterized as the supremum of those p for which the $\mathcal{R}^{\mathrm{cl}}_p$-classes are finite ($\mu$-a.s.), as well as the infimum of p such that $\mathcal{R}^{\mathrm{cl}}_p$ admits a μ-non-null set of points with infinite classes. Much less obvious is the similar characterization of p_u, obtained by O. Häggström and Y. Peres, who showed that after p_c, *there is no spontaneous generation of infinite clusters; all infinite clusters are born simultaneously*: If $p_c < p \leq q$, then μ-a.s. every infinite $\mathcal{R}^{\mathrm{cl}}_q$-class contains an infinite $\mathcal{R}^{\mathrm{cl}}_p$-class [HP99]. This explains that the uniqueness phase is an interval.

$$\begin{aligned} p_c &:= \inf\{p : \text{there is a unique infinite cluster for } \mu_p\} \\ &= \sup\{p : \text{there is not a unique infinite cluster for } \mu_p\} \end{aligned}$$

1.3.e Site Percolation

An invariant site percolation on $\mathcal{G}$ is a probability measure $\mathbf{P}$ on the space $\{0,1\}^{\mathtt{V}}$ that is invariant under a certain group of automorphisms of $\mathcal{G}$. To a site percolation corresponds a bond percolation by the equivariant map $\pi : \{0,1\}^{\mathtt{V}} \to \{0,1\}^{\mathtt{E}}$ sending a coloring of the vertices $\mathtt{V}$ to the coloring of the edges $\mathtt{E}$ where an edge gets color 1 if and only if both its endpoints are colored 1.

1.3.f Graphings

Let $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ be the generating system defining the Cayley graph $\mathcal{G}$ and e_i be the edge $[\rho, \gamma_i\rho]$.

If $\pi(x)(e_i) = 1$, the vertices $\rho, \gamma_i\rho$ are in the same cluster of $\pi(x)$, and $x, \gamma_i^{-1}x$ are $\mathcal{R}^{\mathrm{cl}}$-equivalent. Define the Borel set $A_i := \{x \in X : \pi(x)(e_i) = 1\}$ and the partial Borel isomorphism $\varphi_i = \gamma^{-1}_{i|A_i}$, the restriction of γ_i^{-1} to A_i. The family $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ is a *graphing* (in the sense of [Lev95, Gab00] – see also Section 8) that generates $\mathcal{R}^{\mathrm{cl}}$: the latter is the smallest equivalence relation such that $x \sim \varphi_i(x)$, for every $x \in A_i$. For instance, in the above standard coupling (ex. 1.3.d), the cluster

equivalence relation $\mathcal{R}^{\mathrm{cl}}_p$ is generated by the graphing $\Phi_p = (\varphi_1^p, \varphi_2^p, \cdots, \varphi_n^p)$, where φ_i^p is the restriction of γ_i^{-1} to $A_i^p := \{x \in [0,1]^{\mathtt{E}} : x(e_i) \leq p\}$.

Conversely, given a free p.m.p. Γ-action on (X, μ), consider n Borel subsets A_i, partial isomorphisms $\varphi_i = \gamma^{-1}_{i|A_i}$, the graphing $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ and the generated equivalence relation $\mathcal{R}_\Phi$. The coloring $\pi(x)(e_i) = 1$ iff x belongs to A_i extends by Γ-equivariance to a map $\pi : X \to \{0,1\}^{\mathtt{E}}$ whose cluster equivalence relation $\mathcal{R}^{\mathrm{cl}}$ coincides with $\mathcal{R}_\Phi$.

1.4 Harmonic Dirichlet Functions and Clusters for Cayley Graphs

We are now able to state the main result of this section. Recall that we have at hand: (1) A locally finite graph $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ with a free action of a countable group Γ, transitive on $\mathtt{V}$; (2) A standard probability measure space (X, μ) with a free measure-preserving Γ-action, and with a Γ-equivariant map $\pi : X \to \{0,1\}^{\mathtt{E}}$; (3) The two associated cluster ($\mathcal{R}^{\mathrm{cl}}$) and full ($\mathcal{R}^{\mathrm{fu}}$) equivalence relations.

Theorem 1.5 *For μ-a.e. x in the uniqueness set U^π, the cluster $\mathcal{G}_x$ of the vertex ρ in the colored graph $\pi(x)$ belongs to $\mathcal{O}_{\mathbf{HD}}$ if and only if $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$.*

For the purpose of proving this result, very little has to be known about the L^2 Betti numbers of equivalence relations. Just assume the following "black box", which will be further developed in Section 7. The reader feeling more comfortable with the notion of *cost* may think at first glance that $\beta_1(\mathcal{R}) = \mathrm{cost}(\mathcal{R}) - 1$ (see Section 8, item "cost").

Fact 1. For each measurably defined subrelation $\mathcal{R}$ of $\mathcal{R}^{\mathrm{fu}}$ on a non-null Borel subset Y of X, there is a well-defined notion of first L^2 Betti number $\beta_1(\mathcal{R}, \mu_Y)$, where μ_Y denotes the normalized restricted measure $\frac{\mu_{|Y}}{\mu(Y)}$ ([Gab02]). In particular, $\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu)$, $\beta_1(\mathcal{R}^{\mathrm{fu}}_{|Y}, \mu_Y)$ and $\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y}, \mu_Y)$ are well defined.

Fact 2. If Y meets almost all $\mathcal{R}^{\mathrm{fu}}$-classes, then $\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu) = \mu(Y)\beta_1(\mathcal{R}^{\mathrm{fu}}_{|Y}, \mu_Y)$ [Gab02, Cor. 5.5].

Fact 3. The first L^2 Betti number $\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y}, \mu_Y)$ of $\mathcal{R}^{\mathrm{cl}}_{|Y}$ vanishes if and only if for μ-almost every $y \in Y$, the graph $\mathcal{G}_y$ belongs to $\mathcal{O}_{\mathbf{HD}}$ (Theorem 0.3).

Remark. However, when Y is $\mathcal{R}^{\mathrm{cl}}$-*saturated* (the $\mathcal{R}^{\mathrm{cl}}$-class of every $y \in Y$ is entirely contained in Y), these numbers are "easily explicitly defined": consider the space $\mathbf{HD}(\mathcal{G}_x)$ of harmonic Dirichlet functions on $\mathcal{G}_x := \pi(x)(\rho)$. Its image $d\mathbf{HD}(\mathcal{G}_x)$ by the coboundary operator d in the ℓ^2 cochains $C^1_{(2)}(\mathcal{G}_x)$ is a closed subspace of $C^1_{(2)}(\mathcal{G})$ (every edge outside $\mathcal{G}_x$ is orthogonal to it), isomorphic to $\mathbf{HD}(\mathcal{G}_x)/\mathbb{C}$. Denote by $p_x : C^1_{(2)}(\mathcal{G}) \to d\mathbf{HD}(\mathcal{G}_x)$ the orthogonal projection and, for each edge $e \in \mathtt{E}$, denote by $\mathbf{1}_e$ the characteristic function of the edge e. Let $e_1, e_2, \cdots, e_n$ be a set of orbit representatives for the Γ-action on $\mathtt{E}$.

Proposition 1.6 *Let Y be a non-null* $\mathcal{R}^{\mathrm{cl}}$-saturated *Borel subset of X. The first L^2 Betti number of the restricted equivalence relation $\mathcal{R}^{\mathrm{cl}}_{|Y}$ on (Y, μ_Y) equals*

$$\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y}, \mu_Y) = \frac{1}{\mu(Y)} \sum_{i=1}^{n} \int_Y \langle p_y(\mathbf{1}_{e_i}) | \mathbf{1}_{e_i} \rangle d\mu(y).$$

To prove this, we essentially use Theorem 7.5 stating that $\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y}, \mu_Y) = \dim_{\mathcal{R}^{\mathrm{cl}}_{|Y}} \int_Y d(\mathbf{HD}(\mathcal{G}_y))\, d\mu_Y(y)$ and then the definition of the dimension (see [Gab02, Prop. 3.2 (2)]). See also Proposition 2.7.

PROOF: (of Theorem 1.5) If U^π is a μ-null set, the theorem is empty. Up to replacing X by the union of the Γ-orbits meeting U^π, one may assume that U^π meets every $\mathcal{R}^{\mathrm{fu}}$-class of X. The following are then equivalent:

1. $\mathcal{G}$ is in $\mathcal{O}_{\mathbf{HD}}$
2. $\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu) = 0$
3. $\beta_1(\mathcal{R}^{\mathrm{fu}}_{|U^\pi}, \mu_{U^\pi}) = 0$
4. $\beta_1(\mathcal{R}^{\mathrm{cl}}_{|U^\pi}, \mu_{U^\pi}) = 0$
5. for μ-almost every $x \in U^\pi$, the graph $\mathcal{G}_x$ is in $\mathcal{O}_{\mathbf{HD}}$

The equivalence $1 \iff 2$ follows from fact 3 (i.e. Theorem 0.3) applied to the map π_1 of the example 1.3.a, since in this case $X = Y$, $\mathcal{R}^{\mathrm{cl}} = \mathcal{R}^{\mathrm{fu}}$ and almost every $\mathcal{G}_x$ equals $\mathcal{G}$.

The equivalence $2 \iff 3$ follows from fact 2.

By Proposition 1.4, $\mathcal{R}^{\mathrm{cl}}_{|U^\pi} = \mathcal{R}^{\mathrm{fu}}_{|U^\pi}$. The key point of the proof is that from [Gab02] these numbers depend only on the equivalence relation: one gets $3 \iff 4$. Again, fact 3 shows the equivalence $4 \iff 5$, after noticing that μ and the normalized measure μ_{U^π} are equivalent on U^π.

It remains to move the quantifier (μ-almost every $x \in U^\pi$) outside the equivalence $1 \iff 5$. Let $Y \subset U^\pi$ be the Borel subset of points such that the graph $\mathcal{G}_y$ is in $\mathcal{O}_{\mathbf{HD}}$. If Y is non-null, then the argument applied to Y shows that $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$ and thus $Y = U^\pi$ a.s. This implies that in case $\mathcal{G}$ does not belong to $\mathcal{O}_{\mathbf{HD}}$, then for μ-almost every $x \in U^\pi$, the graph $\mathcal{G}_x$ is not in $\mathcal{O}_{\mathbf{HD}}$. ■

Remark 1.7 *Observe that the freeness of the Γ-action on X is a hypothesis made to simplify some arguments (γ is the unique element of the group sending x to γx) and to apply more directly results from [Gab02]. However, thanks to the example 1.3.c, the above Theorem 1.5 admits a natural generalization without it.*

2 Percolation on Transitive Graphs

Let $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ be a locally finite transitive [18] graph. Let $\mathrm{Aut}(\mathcal{G})$ be the automorphism group of $\mathcal{G}$ with the topology of pointwise convergence. Let H be a closed subgroup of $\mathrm{Aut}(\mathcal{G})$. We assume that H acts transitively on the set $\mathtt{V}$ of vertices. It is locally compact and the stabilizer of each vertex is compact. Let ρ be a base vertex and denote by K_ρ its stabilizer.

The action of H on $\mathtt{E}$ induces an action on the space $\Omega = \{0,1\}^{\mathtt{E}}$ of colorings: for each $h \in H$, $\omega' = h \cdot \omega$ if and only if $\omega'(e) = \omega(h^{-1}e)$ for every edge $e \in \mathtt{E}$.

Let (X, μ) be a standard Borel probability space together with

- a probability measure-preserving (p.m.p.) action of H, which is *essentially free* [19], and
- an H-equivariant Borel map $\pi : X \to \{0,1\}^{\mathtt{E}}$.

The push-forward measure $\pi_*\mu$ is an H-invariant bond percolation on $\mathcal{G}$.

[18] The quasi-transitive case is very similar and we restrict our attention to the transitive one only to avoid an excess of technicality. The modifications for quasi-transitivity are presented in section 5

[19] the Borel set of points $x \in X$ with non-trivial stabilizer have μ-measure 0

2.1 The Full Equivalence Relation

Consider now the space $X \times \mathcal{G}$ with the diagonal action of H. It is an H-equivariant field of graphs above X, all isomorphic to $\mathcal{G}$: $x \mapsto \{x\} \times \mathcal{G}$. It is also a "laminated space", with leaves $\{x\} \times \mathcal{G}$.

Dividing out by the diagonal action of H, one gets the laminated space $\mathcal{L}^{\mathrm{fu}} = H\backslash(X \times \mathcal{G})$: the **full lamination**. It is a (huge, highly disconnected) graph with vertex set $H\backslash(X \times \mathtt{V})$ and edge set $H\backslash(X \times \mathtt{E})$. A **leaf** is a connected component of this graph.

Denote by $X^\bullet$ the image in $\mathcal{L}^{\mathrm{fu}}$ of the space $X \times \{\rho\}$. Because of the transitivity on $\mathtt{V}$ of the H-action, $X \times \{\rho\}$ meets every H-orbit of $X \times \mathtt{V}$, so that $X^\bullet$ equals $H\backslash(X \times \mathtt{V})$:

$$\left(\begin{array}{rcl} X \simeq X \times \{\rho\} & \longrightarrow & X^\bullet = H\backslash(X \times \mathtt{V}) \\ x & \longmapsto & (x,\rho) \sim (hx, h\rho) \end{array} \right) \tag{1}$$

In particular, two points of $X \times \{\rho\}$ happen to be identified in $X^\bullet$, i.e. $(x,\rho) \sim (hx,\rho)$, if and only if h belongs to K_ρ. Thanks to the compactness of the stabilizer K_ρ of ρ, the space $X^\bullet$ gets naturally the structure of a standard Borel space (see Proposition 2.4, Section 2.3).

$$\left(\begin{array}{rcl} X^\bullet = H\backslash(X \times \mathtt{V}) & \simeq & K_\rho\backslash X \\ (hx, h\gamma\rho) & \mapsto & K_\rho \gamma^{-1} x \end{array} \right)$$

Denote by $\mu^\bullet$ the push-forward of the measure μ to $X^\bullet$. Because of the freeness of the H-action on X, the leaf of $\mu^\bullet$-almost every $x^\bullet \in X^\bullet$ is isomorphic to $\mathcal{G}$.

Define the **full equivalence relation** $\mathcal{R}^{\mathrm{fu}}$ on $X^\bullet$ by $x^\bullet \mathcal{R}^{\mathrm{fu}} y^\bullet$ if and only if $x^\bullet$ and $y^\bullet$ are vertices of the same $\mathcal{L}^{\mathrm{fu}}$-leaf.

> $\mathcal{R}^{\mathrm{fu}}$: *Two points $x^\bullet, y^\bullet$ are $\mathcal{R}^{\mathrm{fu}}$-equivalent if and only if they admit two representatives in $X \times \mathtt{V}$ with the same first coordinate, iff they admit two representatives x, y in X for which there exists $h \in H$ such that $hx = y$, iff any of their representatives are in the same H-orbit.*

It inherits naturally an *unoriented graphing* and a *smooth field of graphs* (see Section 8, and examples 7.1, 7.3 of Section 7) from the edge set $H\backslash(X \times \mathtt{E})$, where the graph associated with each point admits an isomorphism with $\mathcal{G}$, canonical up to "rotation around ρ", i.e. up to the action of the stabilizer K_ρ of ρ.

Theorem 2.1 *The equivalence relation $\mathcal{R}^{\mathrm{fu}}$ preserves the measure $\mu^\bullet$ if and only if the group H is unimodular.*

This result is just an application of Theorem 2.5 below. It sheds another light on the *unimodularity assumption* and on the *Mass Transport Principle* (see the proof of Theorem 2.5).

Example 2.2 *The simplest example of the graph $\mathcal{G}$ made of a single infinite line is quite eloquent, with $H = \mathrm{Aut}(\mathcal{G}) \simeq \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$. The compact subgroup $K_\rho = \mathbb{Z}/2\mathbb{Z}$ is finite and $X^\bullet = (\mathbb{Z}/2\mathbb{Z})\backslash X$. If the H-action on X is ergodic, then the lamination $\mathcal{L}^{\mathrm{fu}}$ is not orientable, so that the associated unoriented graphing cannot be made (measurably) oriented.*

2.2 The Cluster Equivalence Relation

Now, thanks to the map $\pi : X \to \{0,1\}^{\mathtt{E}}$, the field of graphs $x \mapsto \{x\} \times \mathcal{G}$ becomes an H-equivariant field of colored graphs $x \mapsto \pi(x)$ so that each leaf of $\mathcal{L}^{\mathrm{fu}}$ becomes a colored graph.

By removing all the 0-colored edges, one defines a subspace $\mathcal{L}^{\mathrm{cl}}$ of $\mathcal{L}^{\mathrm{fu}}$: the **cluster lamination**. A leaf of $\mathcal{L}^{\mathrm{cl}}$ is a connected component of 1-colored (or retained) edges.

Define the **cluster equivalence relation** $\mathcal{R}^{\mathrm{cl}}$ on $X^\bullet$ by $x^\bullet \mathcal{R}^{\mathrm{cl}} y^\bullet$ if and only if $x^\bullet$ and $y^\bullet$ are vertices of the same $\mathcal{L}^{\mathrm{cl}}$-leaf. It is a subrelation of $\mathcal{R}^{\mathrm{fu}}$.

The leaf of $\mu^\bullet$-almost every $x^\bullet$ is a graph $\mathcal{L}^{\mathrm{fu}}_{x^\bullet}$ which admits an isomorphism with the cluster $\mathcal{G}_x := \pi(x)(\rho)$ of the vertex ρ in the subgraph $\pi(x)$ of $\mathcal{G}$ for any representative x of $x^\bullet$ (just observe that since $k\rho = \rho$, for any $k \in K_\rho$, the clusters of ρ for x on the one hand and for kx on the other hand are isomorphic: $\pi(kx)(\rho) = k\pi(x)(\rho)$). Thus the $\mathcal{R}^{\mathrm{cl}}$-class of $x^\bullet$ is infinite if and only if the corresponding clusters $\pi(x)(\rho)$ are infinite. For each $x^\bullet \in X^\bullet$, the family of $\mathcal{R}^{\mathrm{cl}}$-classes into which its $\mathcal{R}^{\mathrm{fu}}$-class decomposes is in bijection with the clusters of $\pi(x)$. The $\mathcal{R}^{\mathrm{fu}}$-class of $x^\bullet$ contains n infinite $\mathcal{R}^{\mathrm{cl}}$-classes iff $\pi(x)$ has n infinite clusters.

$\mathcal{R}^{\mathrm{cl}}$: *Two points $x^\bullet, y^\bullet$ are $\mathcal{R}^{\mathrm{cl}}$-equivalent if and only if they admit two representatives in $X \times \mathtt{V}$ with the same first coordinate x and second coordinates in the same connected component of $\pi(x)$, iff they admit two representatives x, y in X for which there exists $h \in H$ such that $hx = y$ and the vertices $\rho, h^{-1}\rho$ are in the same cluster of $\pi(x)$.*

Let's check by hand that the above characterization doesn't depend on the choice of representatives. This h defines an isomorphism between the cluster $\pi(x)(\rho) = \pi(x)(h^{-1}\rho)$ and $\pi(hx)(hh^{-1}\rho) = \pi(y)(\rho)$. If k_1x and k_2y are two other representatives, $k_1, k_2 \in K_\rho$, then $k_2hk_1^{-1}(k_1x) = k_2y$ (i.e. h has to be replaced by $k_2hk_1^{-1}$), then $k_1\rho = \rho$ and $k_1h^{-1}\rho = k_1h^{-1}k_2^{-1}\rho = (k_2hk_1^{-1})^{-1}\rho$ are in the same cluster of $\pi(k_1(x))$.

The equivalence relation $\mathcal{R}^{\mathrm{cl}}$ inherits naturally an *unoriented graphing* and a *smooth field of graphs* (see Section 8, and examples 7.1, 7.3 of Section 7).

Remark 2.3 *Let $Y^\bullet \subset X^\bullet$ be the union of the infinite $\mathcal{R}^{\mathrm{cl}}$-classes. Assume the action of H on X is ergodic. Then the invariant percolation $\pi_*\mu$ has indistinguishable infinite clusters in the sense of [LS99, Sect. 3] if and only if the restriction $\mathcal{R}^{\mathrm{cl}}_{|Y}$ is ergodic.*

2.3 Measure Invariance, Unimodularity and the Mass-Transport Principle

Recall that a locally compact second countable group G admits a left-invariant Radon measure, its *Haar measure* m, unique up to a multiplicative constant. Pushed forward by right-multiplication, the measure is again left-invariant, and thus proportional to m. One gets a homomorphism $\mathrm{mod} : G \to \mathbb{R}^*_+$, the **modular map**, which encodes the defect for m to be also right-invariant. In case the modular map is trivial ($\mathrm{mod}(G) = \{1\}$), i.e. m is also right-invariant, then the group G is called **unimodular**.

Let (X, μ) be a standard Borel space with a probability measure and an essentially free measure-preserving action of a locally compact second countable group G. Let K be a compact open subgroup of G. Restricted to K, the modular function is trivial.

Proposition 2.4 *The space $\overline{X} = K\backslash X$ is a standard Borel space. The quotient map $X \to K\backslash X$ admits a Borel section.*

PROOF: The following argument has been explained to me by A. Kechris. Pushing forward the normalized Haar measure m on K by the Borel map (for any fixed x) $K \to X$, $k \mapsto kx$ defines a measure m_x and thus a Borel map from X to the standard Borel space of probability measures on X ([Kec95, Th. 17.25]). But the right invariance of m on K shows [20] that x and y are in the same K-orbit iff $m_x = m_y$. The K-action is then smooth. It follows from [Kec95, Ex. 18.20] that the action has a Borel selector. ■

Let $\overline{\mathcal{R}}$ be the **reduced** equivalence relation defined on $\overline{X}$ by $\overline{x}\overline{\mathcal{R}}\overline{y}$ iff $\overline{x}$ and $\overline{y}$ admit G-equivalent preimages. Let's denote by $\overline{\mu}$ the push-forward probability measure on $\overline{X} = K\backslash X$, or by $\overline{\mu}_K$ if one wants to emphasize the choice of K. Section 8 recalls the terminology for the next Theorem.

Theorem 2.5 *The equivalence relation $\overline{\mathcal{R}}$ on $\overline{X}$ is standard countable. It preserves the measure $\overline{\mu}_K$ if and only if G is unimodular.*

PROOF: It is obviously a Borel subset of $\overline{X} \times \overline{X}$. The countability of the classes comes from that of the set $K\backslash G$.

The statement about unimodularity is quite natural once one realizes that the decomposition of μ relatively to $\overline{\mu}$ makes use of the right invariant Haar measure on G. However, we will follow elementary but enlightening facts leading by two ways to the result.

Recall (see Section 8) that $\overline{\mathcal{R}}$ preserves $\overline{\mu}_K$ if and only if the measures ν_1 and ν_2 on the set $\overline{\mathcal{R}} \subset X \times X$ coincide, defined with respect to the projections on the first (resp. second) coordinate pr_1 (resp. pr_2) by $\nu_1(C) = \int_X \#(C \cap pr_1^{-1}(x))d\overline{\mu}_K(x)$ and $\nu_2(C) = \int_X \#(C \cap pr_2^{-1}(y))d\overline{\mu}_K(y)$.

Proof by hand: If K, K' are compact open subgroups of G, then K is made of unimodular elements of G. The intersection $K \cap K'$ is a compact open subgroup of G. By a covering argument, its index in K is finite: $[K : K \cap K'] := \#K/(K \cap K') = \frac{m(K)}{m(K \cap K')}$. For $\gamma \in G$, $m(\gamma^{-1}K\gamma) = \mathrm{mod}(\gamma)m(K)$. In particular, G is unimodular iff all the conjugates of K have the same Haar measure. Observe that $[K : \gamma K\gamma^{-1} \cap K] = \mathrm{mod}(\gamma)[K : K \cap \gamma^{-1}K\gamma]$ and that K and K' as well as their Haar measures are commensurable, so that the modular function on G is rational. The reduction map $(K \cap K')\backslash X \to K\backslash X$ is a.s. $[K : K \cap K']$-to-one, and yields a disintegration of the push-forward measure $\overline{\mu}_{K\cap K'}$ with respect to $\overline{\mu}_K$, with normalized counting measure in the fibers.
For $\gamma \in G$, consider the graph $C_\gamma := \{(x, \gamma x) : x \in X\}$ and its image in $\overline{\mathcal{R}}$:

$$\overline{C}_\gamma := \{(\overline{x}, \overline{\gamma x}) : x \in X\} = \{([K.x], [K.\gamma x]) : x \in X\}.$$

For every $k \in K$, $\overline{C}_\gamma = \overline{C}_{k\gamma}$. Two points (x_1, y_1) and (x_2, y_2) of the "curve" C_γ define the same point in $\overline{C}_\gamma$ *iff* there exist $k, k' \in K$ such that $x_1 = kx_2$ and $\gamma x_1 = y_1 = k'y_2 = k'\gamma x_2$, *iff* (since by freeness of the action $k = \gamma^{-1}k'\gamma$) there exists $k \in K \cap (\gamma^{-1}K\gamma)$ such that $x_1 = kx_2$, $\gamma x_1 = y_1$ and $\gamma x_2 = y_2$ *iff* there exists $k' \in (\gamma K\gamma^{-1}) \cap K$ such that $x_1 = \gamma^{-1}y_1$, $x_2 = \gamma^{-1}y_2$ and $y_1 = k'y_2$. Thus, the preimage in C_γ (not in $X \times X$!) of a point in $\overline{C}_\gamma$ is of the form (for certain x and y): $\{(kx, \gamma kx) : k \in (K \cap \gamma^{-1}K\gamma)\} = \{\gamma^{-1}k'y, k'y) : k' \in (\gamma K\gamma^{-1} \cap K)\}$, so that

$$\overline{C}_\gamma \simeq (K \cap \gamma^{-1}K\gamma)\backslash X \qquad \text{or} \qquad \overline{C}_\gamma \simeq (\gamma K\gamma^{-1} \cap K)\backslash X,$$

according to whether $\overline{C}_\gamma$ is parameterized by its first coordinate or its second coordinate. We have thus proved the following:

[20] $m_{hx}(A) = m(\{k : khx \in A\}) = m(\{k'h^{-1} : k'x \in A\}) = m(\{k : kx \in A\}) = m_x(A)$

Proposition 2.6 *The projection of $\overline{C}_\gamma \subset \overline{\mathcal{R}}$ to the first coordinate is $[K : K \cap \gamma^{-1}K\gamma]$-to-one. To the second coordinate, it is $[K : \gamma K \gamma^{-1} \cap K]$-to-one.*

It follows that the measures ν_1 and ν_2 restricted to $\overline{C}_\gamma$ satisfy $\nu_2 = \frac{[K:\gamma K\gamma^{-1}\cap K]}{[K:K\cap\gamma^{-1}K\gamma]}\,\nu_1 = \mathrm{mod}(\gamma)\,\nu_1$.

Proof by Mass-Transport Principle: Denote by W the countable discrete set G/K and observe that $\overline{X}$ is isomorphic with the quotient $G\backslash(X \times W)$ (where the G-action is diagonal).

$$\begin{pmatrix} G\backslash(X\times W) & \simeq & K\backslash X = \overline{X} \\ (gx, g\gamma K) & \mapsto & K\gamma^{-1}x \end{pmatrix}$$

Denote by ρ the class $K \in G/K$ and by K_v the stabilizer of $v \in W$ in the left multiplication G-action. In particular, $K_\rho = K$.

Two points $\bar{x}_1$ and $\bar{x}_2$ of $\overline{X}$ are $\overline{\mathcal{R}}$-equivalent iff they admit representatives in $X \times W$ with the same first coordinate. One thus gets an identification of $\overline{\mathcal{R}}$ with $G\backslash(X \times W \times W)$ (where the G-action is diagonal on the three coordinates), thanks to the two coordinate-forgetting projections (where W_1, W_2 are two copies of W):

$$\begin{tikzcd}[column sep=small] & G\backslash(X\times W_1)=\overline{X} \\ G\backslash(X\times W_1\times W_2) \arrow[ur, "pr_1"] \arrow[dr, "pr_2"'] & \\ & G\backslash(X\times W_2)=\overline{X} \end{tikzcd}$$

It becomes equivalent to consider a function F on $\overline{\mathcal{R}}$ or a G-invariant function f on $X \times W \times W$. Thus, for non-negative functions

$$\begin{aligned} \nu_1(F) &= \int_{\overline{\mathcal{R}}} F(\bar{x}_1,\bar{x}_2)d\nu_1 = \int_{\overline{X}} \sum_{\bar{x}_2\sim\bar{x}_1} F(\bar{x}_1,\bar{x}_2)d\bar{\mu}(\bar{x}_1) \\ &= \int_X \sum_{v_2\in W} f(x,\rho,v_2)d\mu(x) \end{aligned}$$

while

$$\nu_2(F) = \int_X \sum_{v_1\in W} f(x,v_1,\rho)d\mu(x)$$

On the other hand, the *mass-transport principle* below essentially gives the correcting terms for ν_1 and ν_2 to coincide. In particular, unimodularity, equivalent to the coincidence of the Haar measures $m(K_{v_1}) = m(K_\rho)$ for every v_1, is equivalent to the preservation for $\overline{\mathcal{R}}$ of the measure $\bar{\mu}$.

The **mass-transport principle**:

$$\int_X \sum_{v_2\in W} f(x,\rho,v_2)\ m(K_\rho)\ d\mu(x) = \int_X \sum_{v_1\in W} f(x,v_1,\rho)\ m(K_{v_1})\ d\mu(x),$$

where m is the Haar measure on G, is a useful device in invariant percolation theory. For details, see [BLPS99a] where I took the following two-line proof, credited to W. Woess. Let $\bar{f}(v,v') := \int_X f(x,v,v')\ d\mu(x)$ denote the mean value.

$$\begin{aligned} \sum_{v_2\in W} \bar{f}(\rho,v_2)\ m(K_\rho) &= \sum_{v_2\in W} \bar{f}(\rho,v_2)\ m(\{g : g\rho = v_2\}) &= \int_G \bar{f}(\rho,g\rho)\ dm(g) \\ \sum_{v_1\in W} \bar{f}(v_1,\rho)m(K_{v_1}) &= \sum_{v_1\in W} \bar{f}(v_1,\rho)m(\{\underbrace{g : gv_1 = \rho}_{\{g:g^{-1}\rho=v_1\}}\}) &= \int_G \bar{f}(g^{-1}\rho,\rho)dm(g) \end{aligned}$$

And, the last terms are equal, thanks to the G-invariance of f and μ. ∎

2.4 Some computation

Assume that H is transitive and unimodular.
Recall that $\mathcal{G}_x$ denotes, for $x \in X$, the cluster of the vertex ρ in the subgraph $\pi(x)$. Let $p_x : C^1_{(2)}(\mathcal{G}) \to d\mathbf{HD}(\mathcal{G}_x)$ be the orthogonal projection from the space of ℓ^2 cochains of $\mathcal{G}$ to the image, under the coboundary d, of $\mathbf{HD}(\mathcal{G}_x)$ in $C^1_{(2)}(\mathcal{G})$. Denote by $\mathbf{1}_{e_1}, \mathbf{1}_{e_2}, \cdots, \mathbf{1}_{e_n} \in C^1_{(2)}(\mathcal{G})$ the characteristic functions of the edges $e_1, e_2, \cdots, e_n$ adjacent to the base point ρ.

Proposition 2.7 *Let $Y^\bullet$ be a non-null $\mathcal{R}^{\mathrm{cl}}$-saturated Borel subset of $X^\bullet$, and Y its preimage in X. The first L^2 Betti number of the restricted measure equivalence relation $\mathcal{R}^{\mathrm{cl}}_{|Y^\bullet}$ on $(Y^\bullet, \mu^\bullet_{Y^\bullet})$ equals*

$$\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y^\bullet}, \mu^\bullet_{Y^\bullet}) = \frac{1}{2\mu(Y)} \sum_{i=1}^{n} \int_Y \langle p_y(\mathbf{1}_{e_i}) | \mathbf{1}_{e_i} \rangle d\mu(y).$$

Here, $\mu^\bullet_{Y^\bullet}$ is of course the normalized restriction of $\mu^\bullet$ to $Y^\bullet$. The $\frac{1}{2}$ terms just reflects that, the graph being transitive, each edge is counted twice: once per endpoint, while $\frac{1}{\mu(Y)}$ is just designed to normalize.

PROOF: We use first Theorem 7.5 stating that $\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y^\bullet}, \mu_{Y^\bullet}) = \dim_{\mathcal{R}^{\mathrm{cl}}_{|Y^\bullet}} \int_{Y^\bullet} d(\mathbf{HD}(\mathcal{G}_{y^\bullet}))\, d\mu^\bullet_{Y^\bullet}(y^\bullet)$; second the definition of the dimension (see [Gab02, Prop. 3.2 (2)]): A measurable labeling (see Proposition 2.4) $e^\bullet_1, e^\bullet_2, \cdots, e^\bullet_n$ of the edges adjacent to $y^\bullet$ leads to measurable vector fields $y^\bullet \mapsto \mathbf{1}_{e^\bullet_i} \in C^1_{(2)}(\mathcal{G}_{y^\bullet})$ that define a family of fields of representative (in the sense of [Gab02]), except that each edge is represented twice (the additionnal difficulty of a possible loop in $\mathcal{G}$ is dismissed by the fact that it would give a vector orthogonal to $d(\mathbf{HD}(\mathcal{G}_{y^\bullet}))$. Third, we use the relation between the objects with and without a $\bullet$ sign. ■

Remark 2.8 *In case $Y^\bullet$ is not $\mathcal{R}^{\mathrm{cl}}$-saturated, the families of fields of representative are more delicate to describe. However, Corollary 5.5 of [Gab02], for induction on Borel subsets (see fact 2, Subsection 1.4), applied to $Y^\bullet$ and its $\mathcal{R}^{\mathrm{cl}}$-saturation leads to the same formula except that the domain of integration is now the H-saturation HY of Y:*

$$\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y^\bullet}, \mu^\bullet_{Y^\bullet}) = \frac{1}{2\mu(Y)} \sum_{i=1}^{n} \int_{HY} \langle p_y(\mathbf{1}_{e_i}) | \mathbf{1}_{e_i} \rangle d\mu(y).$$

Observe that the quantity on the right in the above Proposition 2.7 in fact only depends on the image $\Omega_Y := \pi(Y)$ in $\Omega = \{0,1\}^{\mathtt{E}}$

$$\beta_1(\mathcal{R}^{\mathrm{cl}}_{|Y^\bullet}, \mu^\bullet_{Y^\bullet}) = \frac{1}{2\pi_*\mu(\Omega_Y)} \sum_{i=1}^{n} \int_{\Omega_Y} \langle p_\omega(\mathbf{1}_{e_i}) | \mathbf{1}_{e_i} \rangle d\pi_*\mu(\omega).$$

Concerning the full equivalence relation, one gets,

$$\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu^\bullet) = \frac{1}{2} \sum_{i=1}^{n} \int_X \langle p_x(\mathbf{1}_{e_i}) | \mathbf{1}_{e_i} \rangle d\mu(x).$$

But in the case of $\mathcal{R}^{\mathrm{fu}}$, the projection p_x doesn't depend on x: it is just the projection

$$p : C^1_{(2)}(\mathcal{G}) \to d\mathbf{HD}(\mathcal{G}) \simeq \mathbf{HD}(\mathcal{G})/\mathbb{C}.$$

It follows that

Proposition 2.9 *For the full equivalence relation on $X^\bullet$:*

$$\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu^\bullet) = \frac{1}{2}\sum_{i=1}^{n} \langle p(\mathbf{1}_{e_i}) | \mathbf{1}_{e_i}\rangle. \tag{2}$$

Observe that this quantity doesn't even depend on what happens on X nor on the choice of H, once H is unimodular and transitive on the vertices: It is an invariant of the graph.

Definition 2.10 *Call it the* **first ℓ^2 Betti number** *of $\mathcal{G}$ and denote it*

$$\beta_1(\mathcal{G}) := \frac{1}{2}\sum_{i=1}^{n} \langle p(\mathbf{1}_{e_i}) | \mathbf{1}_{e_i}\rangle. \tag{3}$$

It is clear[21] that $\beta_1(\mathcal{G}) = 0$ if and only if $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$.

3 Harmonic Dirichlet Functions and Clusters for Transitive Graphs

In this section, we give the proof of Theorem 0.4 of the introduction by putting/proving it in the more general context of selectability.

Recall that we have at hand: (1) A locally finite graph $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ with a transitive (on $\mathtt{V}$) action of a closed group H of automorphisms; (2) An H-invariant probability measure $\mathbf{P}$ on the set $\Omega = \{0,1\}^{\mathtt{E}}$ of colorings of $\mathcal{G}$.

3.1 The action made free

The H-action on Ω being not necessarily free, let's consider a diagonal H-action on $X = \Omega \times Z$, where Z is a standard Borel space with an essentially free probability measure-preserving action of H. An example of such a Z is furnished by the Lemma 3.2 below.

The diagonal action preserves the product measure μ and is (essentially) free. The obvious projection $\pi : X = \Omega \times Z \to \Omega$ sends μ to $\mathbf{P}$ and is H-equivariant, so that we are in the context of Section 2.

Remark 3.1 *We could probably avoid the detour by the freeness of the action by defining L^2 Betti numbers for groupoids instead of just for equivalence relations, as suggested in [Gab02, p.103]. Notice that such a study of L^2 Betti numbers for measured groupoids has been carried out by R. Sauer (see [Sau03]), using Lück's approach of ℓ^2 theory.*

Lemma 3.2 *If H acts continuously[22] faithfully on a discrete countable set V, then the diagonal action of H on $\check{\Omega} = \left(\{0,1\}^V\right)^{\mathbb{N}}$ is continuous, preserves the Bernoulli measure (product of equiprobabilities on $\{0,1\}$), and is essentially free.*

PROOF: Enumerate the elements of V: $v_1, v_2, \ldots, v_n, \ldots$ and denote by $\check{\Omega}_{i,j}$ the subset of points of $\check{\Omega}$ that are fixed by an element of H which sends v_i to v_j. This subset satisfying infinitely many equations: $\omega(v_i, l) = \omega(v_j, l)$ for each coordinate $l \in \mathbb{N}$, has thus measure 0. The set of points ω with a non-trivial stabilizer is contained in the countable union of the $\check{\Omega}_{i,j}$; it has measure zero. ■

[21] $\langle p(\mathbf{1}_{e_i})|\mathbf{1}_{e_i}\rangle = \langle p^2(\mathbf{1}_{e_i})|\mathbf{1}_{e_i}\rangle = \langle p(\mathbf{1}_{e_i})|p(\mathbf{1}_{e_i})\rangle = 0$ iff $h.p(\mathbf{1}_{e_i}) = p(h.\mathbf{1}_{e_i}) = 0$ for all $h \in H$ and $i = 1, \cdots, n$.

[22] The stabilizer of a point of V is a closed open subgroup of H.

3.2 Selectability

Let $\mathcal{G}$ be a locally finite transitive graph and H a closed transitive subgroup of $\mathrm{Aut}(\mathcal{G})$. Equip $\{0,1\}^{\mathtt{V}}$ with the natural action of H induced by its action on $\mathtt{V}$.

Definition 3.3 *Let* $\mathbf{P}$ *be an* H*-invariant percolation on* $\mathcal{G}$*. Let* Δ *be a closed subgroup of* H*.*

A Δ*-equivariant* ***selected cluster*** *on a* Δ*-invariant* $\mathbf{P}$*-non-null Borel subset* $D \subset \{0,1\}^{\mathtt{E}}$ *is a* Δ*-equivariant measurable map* $c : D \to \{0,1\}^{\mathtt{V}}$*, such that* $c(\omega)$ *is the (characteristic function of the) vertex set of one cluster* $C(\omega)$ *of* ω*.*

A Δ*-equivariant* ***virtually selected cluster*** *on a* Δ*-invariant* $\mathbf{P}$*-non-null Borel subset* $D \subset \{0,1\}^{\mathtt{E}}$ *is a* Δ*-equivariant measurable map* $c : D \to \{0,1\}^{\mathtt{V}}$ *such that* $c(\omega)$ *is the (characteristic function of the) vertex set of (the union of) finitely many clusters* $C_1(\omega), C_2(\omega), \cdots, C_{n(\omega)}(\omega)$ *of* ω*.*

Example 3.4 *If almost every subgraph* ω *has a unique infinite cluster, assigning to* ω *this infinite cluster defines an* H*-equivariant selected cluster. And similarly, if almost every subgraph* ω *has finitely many infinite clusters, assigning to* ω *these infinite clusters defines an* H*-equivariant virtually selected cluster.*

Let (X,μ) be a standard Borel probability space together with

- a probability measure-preserving (p.m.p.) action of H, which is free, and
- an H-equivariant Borel map (field of graphs) $\pi : X \to \{0,1\}^{\mathtt{E}}$

so that our situation fits in the general context of Section 2.

Proposition 3.5 *The following are equivalent:*
(1) The invariant percolation $\pi_*(\mu)$ *admits a* H*-equivariant selected cluster,*
(2) There is a non-null Borel subset $T^\bullet$ *of* $X^\bullet$ *to which the restrictions of* $\mathcal{R}^{\mathrm{fu}}$ *and* $\mathcal{R}^{\mathrm{cl}}$ *coincide:* $\mathcal{R}^{\mathrm{fu}}_{|T^\bullet} = \mathcal{R}^{\mathrm{cl}}_{|T^\bullet}$*.*

Moreover, $T^\bullet$ *can be taken to be the image in* $X^\bullet$ *of the set of those* $x \in X$ *whose selected cluster contains the base point* ρ*.*

PROOF: The selected cluster $c : \{0,1\}^{\mathtt{E}} \to \{0,1\}^{\mathtt{V}}$ selects by composition by π one connected component of the splitting of each $\mathcal{L}^{\mathrm{fu}}$-leaf into its $\mathcal{L}^{\mathrm{cl}}$-components. The union of these selected leaves intersects the transversal $X^\bullet$ along a Borel subset $T^\bullet$ which is characterized as the image in $X^\bullet$ of the set of $x \in X$ such that the selected cluster $c(\pi(x))$ contains the base point ρ. Two $\mathcal{R}^{\mathrm{fu}}$-equivalent points in $T^\bullet$ belong to the same $\mathcal{L}^{\mathrm{fu}}$-leaf and both belong to THE selected $\mathcal{L}^{\mathrm{cl}}$-leaf; they are thus $\mathcal{R}^{\mathrm{cl}}$-equivalent.

Conversely, if $T^\bullet$ is a Borel subset of $X^\bullet$ to which the restrictions of $\mathcal{R}^{\mathrm{fu}}$ and $\mathcal{R}^{\mathrm{cl}}$ coincide, then it selects an $\mathcal{L}^{\mathrm{cl}}$-leaf in each $\mathcal{L}^{\mathrm{fu}}$-leaf meeting $T^\bullet$. One can assume that $T^\bullet$ is $\mathcal{R}^{\mathrm{cl}}$-saturated (two points that are $\mathcal{R}^{\mathrm{fu}}$-equivalent and $\mathcal{R}^{\mathrm{cl}}$-equivalent to some point in $T^\bullet$ have to be $\mathcal{R}^{\mathrm{cl}}$-equivalent). Let $\check{T}$ be the preimage of $T^\bullet$ in $X \times \mathtt{V}$. It is an H-invariant subset, whose projection T in X is non-null and H-invariant, and whose intersection with each fiber $(\{x\} \times \mathtt{V}) \cap \check{T}$ is a cluster $\check{c}(x)$ of $\pi(x)$. This defines an H-equivariant map $\check{c} : T \to \{0,1\}^{\mathtt{V}}$. Now the set $(X \times \{\rho\}) \cap \check{T}$ once projected in X corresponds to those $x \in T$ for which $\check{c}(x)$ is the cluster of the base point ρ. This shows that the cluster $\check{c}(x)$ only depends on $\pi(x)$. Moreover, $\pi(T)$ is non-null for the measure $\pi_*\mu$, so that the map $\check{c}$ induces an H-equivariant selected cluster on $\pi(T) \subset \{0,1\}^{\mathtt{E}}$. ■

Recall that a subrelation $\mathcal{S}$ has *finite index* in $\mathcal{R}$ if each $\mathcal{R}$-class splits into finitely many $\mathcal{S}$-classes. The same kind of argument as above shows:

Proposition 3.6 *The following are equivalent:*
(1) The invariant percolation $\pi_*(\mu)$ admits a H-equivariant virtually selected cluster,
(2) There is a non-null Borel subset $T^\bullet$ of $X^\bullet$ to which the restriction of $\mathcal{R}^{\mathrm{cl}}$ has finite index in the restriction of $\mathcal{R}^{\mathrm{fu}}$: $[\mathcal{R}^{\mathrm{fu}}_{|T^\bullet} : \mathcal{R}^{\mathrm{cl}}_{|T^\bullet}] < \infty$.

Moreover, $T^\bullet$ can be taken to be the image in $X^\bullet$ of the set of those $x \in X$ for which one of the selected clusters contains the base point ρ.

Observe that if $\mathcal{G}$ is not a finite graph, then $\mathcal{R}^{\mathrm{fu}}$ has infinite classes and the H-equivariant virtually selected clusters are (almost) all infinite.

Remark 3.7 *The main result of [LS99] (indistinguishability of the infinite clusters) implies that Bernoulli percolation in the nonuniqueness phase admits no H-equivariant virtually selected clusters.*

Remark 3.8 Let Δ be a closed subgroup of H that contains the stabilizer K_ρ of ρ. One can define a notion of full Δ-equivalence relation $\mathcal{R}^{\mathrm{fu}}_\Delta \subset \mathcal{R}^{\mathrm{fu}}$ and cluster Δ-equivalence relation: the intersection $\mathcal{R}^{\mathrm{cl}}_\Delta = \mathcal{R}^{\mathrm{cl}} \cap \mathcal{R}^{\mathrm{fu}}_\Delta$.

$\mathcal{R}^{\mathrm{fu}}_\Delta$: *Two points $x^\bullet, y^\bullet$ are $\mathcal{R}^{\mathrm{fu}}_\Delta$-equivalent if and only if they admit two representatives x, y in X for which there exists $\delta \in \Delta$ such that $\delta x = y$, iff any of their representatives are in the same Δ-orbit.*

Similarly, for the cluster Δ-equivalence relation:

$\mathcal{R}^{\mathrm{cl}}_\Delta$: *Two points $x^\bullet, y^\bullet$ are $\mathcal{R}^{\mathrm{cl}}_\Delta$-equivalent if and only if they admit two representatives x, y in X for which there exists $\delta \in \Delta$ such that $\delta x = y$ and the vertices $\rho, \delta^{-1}\rho$ are in the same cluster of $\pi(x)$.*

In case $\mathcal{G}$ is the Cayley graph of a discrete group Γ (i.e. $K_\rho = \{1\}$ and $X^\bullet = X$) then $\mathcal{R}^{\mathrm{fu}}_\Delta$ is just the equivalence relation defined by the Δ-action on X, while $\mathcal{R}^{\mathrm{cl}}_\Delta = \mathcal{R}^{\mathrm{cl}} \cap \mathcal{R}^{\mathrm{fu}}_\Delta$ is just defined by: $(x, y) \in \mathcal{R}^{\mathrm{cl}}_\Delta$ iff there exists $\delta \in \Delta$ such that $\delta x = y$ and the vertices $\rho, \delta^{-1}\rho$ are in the same cluster of $\pi(x)$.

Exactly along the same arguments as above, one can show that the following are equivalent:
(1) The invariant percolation $\pi_*(\mu)$ admits a Δ-equivariant selected cluster,
(2) There is a non-null Borel subset $T^\bullet$ of $X^\bullet$ to which the restrictions of $\mathcal{R}_\Delta$ and $\mathcal{R}^{\mathrm{cl}}_\Delta$ coincide: $\mathcal{R}^{\mathrm{fu}}_{\Delta|T} = \mathcal{R}^{\mathrm{cl}}_{\Delta|T}$.
And similarly, with finite index, for the virtual notion.

The lamination interpretation of these equivalence relations goes as follows:
Consider first the space $X \times \mathcal{G}$ and divide out by Δ to get the laminated space $\mathcal{L}^{\mathrm{fu}}_\Delta$. Consider now the transversal $\Delta\backslash(X \times \Delta\rho) \subset \Delta\backslash(X \times \mathtt{V})$, which is naturally isomorphic with $X^\bullet = K_\rho\backslash X$ since Δ contains K_ρ, and the equivalence relation defined on it by "belonging to the same $\mathcal{L}^{\mathrm{fu}}_\Delta$-leaf". This is the full Δ-equivalence relation $\mathcal{R}^{\mathrm{fu}}_\Delta \subset \mathcal{R}^{\mathrm{fu}}$ and it appears as the image in $X^\bullet = H\backslash(X \times \mathtt{V})$ of the equivalence relation defined by the Δ-action on X. Just like in Section 1, use now π to get a coloring on the leaves. Define $\mathcal{L}^{\mathrm{cl}}_\Delta$ as the sub-laminated space where the 0-colored edges are removed and $\mathcal{R}^{\mathrm{cl}}_\Delta$ as the subrelation of $\mathcal{R}^{\mathrm{fu}}_\Delta$ induced on $\Delta\backslash(X \times \Delta\rho)$ by "belonging to the same $\mathcal{L}^{\mathrm{cl}}_\Delta$-leaf".

3.3 Selected Clusters and Harmonic Dirichlet Functions

The connections between selected clusters and harmonic Dirichlet functions is very simple:

Theorem 3.9 *Assume $\pi_*\mu$ admits an H-equivariant virtually selected cluster. Assume that the closed subgroup H is unimodular. If $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$, then μ-a.e. virtually selected cluster belongs to $\mathcal{O}_{\mathbf{HD}}$. If $\mathcal{G}$ doesn't belong to $\mathcal{O}_{\mathbf{HD}}$, then μ-a.e. virtually selected cluster doesn't belong to $\mathcal{O}_{\mathbf{HD}}$.*

PROOF: Thanks to unimodularity, the associated equivalence relations $\mathcal{R}^{\mathrm{fu}}$ and $\mathcal{R}^{\mathrm{cl}}$ are measure-preserving (Th. 2.5).

Start with the case of a selected cluster. Denote by c the H-equivariant selected cluster $c : \{0,1\}^{\mathtt{E}} \to \{0,1\}^{\mathtt{V}}$ and $C = c \circ \pi : X \to \{0,1\}^{\mathtt{V}}$. Let T be the Borel subset of $x \in X$ where the selected cluster $C(x)$ contains ρ and let $T^\bullet$ be its image in $X^\bullet$.

The following are equivalent:

1. $\mathcal{G}$ is in $\mathcal{O}_{\mathbf{HD}}$
2. $\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu^\bullet) = 0$
3. $\beta_1(\mathcal{R}^{\mathrm{fu}}_{|T^\bullet}, \mu^\bullet_{T^\bullet}) = 0$
4. $\beta_1(\mathcal{R}^{\mathrm{cl}}_{|T^\bullet}, \mu^\bullet_{T^\bullet}) = 0$
5. for $\mu^\bullet$-almost every $x^\bullet \in T^\bullet$, the graph $\mathcal{L}^{\mathrm{cl}}_{x^\bullet}$ is in $\mathcal{O}_{\mathbf{HD}}$
6. for μ-almost every $x \in T$, the selected cluster $C(x)$ is in $\mathcal{O}_{\mathbf{HD}}$

Theorem 0.3 applied to the field of graphs $x^\bullet \mapsto \mathcal{L}^{\mathrm{fu}}_{x^\bullet} \simeq \mathcal{G}$ (example 7.3) gives the equivalence $1 \iff 2$. When applied to the restriction of that field to $T^\bullet$ (example 7.4), it gives $1 \iff 3$. Observe that the equivalence $2 \iff 3$ is also an application of [Gab02, Cor. 5.5]. Since β_1 is an invariant of the equivalence relation, $3 \iff 4$ is deduced from the coincidence $\mathcal{R}^{\mathrm{cl}}_{|T^\bullet} = \mathcal{R}^{\mathrm{fu}}_{|T^\bullet}$ (prop. 3.5). Theorem 0.3, applied to the field of graphs $x^\bullet \mapsto \mathcal{L}^{\mathrm{cl}}_{x^\bullet}$ (example 7.3) restricted to $T^\bullet$, shows the equivalence $4 \iff 5$. Each $\mathcal{L}^{\mathrm{cl}}_{x^\bullet}$ being isomorphic to the cluster $\pi(x)(\rho) = C(x)$ of any of its representatives, and $\mu^\bullet$ being the push-forward of μ, one deduces the last equivalence.

Let $T' \subset T$ be the Borel subset of points such that the selected cluster is in $\mathcal{O}_{\mathbf{HD}}$. If T' is non-null, then the above arguments applied to T' shows that $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$ and thus $T' = T$ a.s. This implies that in case $\mathcal{G}$ does not belong to $\mathcal{O}_{\mathbf{HD}}$, then for μ-almost every $x \in T$, the selected cluster $C(x)$ is not in $\mathcal{O}_{\mathbf{HD}}$.

For the case of a virtually selected cluster, partition first T as $\coprod T_n$ according to how many clusters are selected. On $T^\bullet_n$ the $\mathcal{R}^{\mathrm{fu}}_{|T^\bullet}$-classes decompose into n $\mathcal{R}^{\mathrm{cl}}_{|T^\bullet}$-classes (prop. 3.6). Then just replace the argument in the proof of the equivalence $3 \iff 4$ above by Proposition 5.11 of [Gab02], asserting that $\beta_1(\mathcal{R}^{\mathrm{cl}}_{|T^\bullet}) = n\beta_1(\mathcal{R}^{\mathrm{fu}}_{|T^\bullet})$. ■

4 Nonuniqueness Phase and Harmonic Dirichlet Functions

This section is concerned with a comparison between two invariant percolations. Its main goal is to prove Theorem 4.2, which implies both Theorem 0.6 (Corollary 4.5) and Theorem 0.7 (Corollary 4.7) of the introduction.

Consider a unimodular transitive group H of automorphisms of $\mathcal{G}$ and two H-invariant percolations μ_1 and μ_2 on $\mathcal{G}$. Recall that given two H-invariant percolations on $\mathcal{G}$, an **H-equivariant coupling** is a p.m.p. H-action on a standard probability measure space (X, μ) with two H-equivariant maps

$$
\begin{array}{ccc}
 & (X,\mu) & \\
{}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\
(\{0,1\}^{\mathtt{E}},\mu_1) & & (\{0,1\}^{\mathtt{E}},\mu_2)
\end{array}
$$

pushing μ to μ_i respectively, i.e. $\pi_{1*}\mu = \mu_1$ and $\pi_{2*}\mu = \mu_2$.

To π_1 and π_2 correspond two laminations $\mathcal{L}_1^{\mathrm{cl}}$ and $\mathcal{L}_2^{\mathrm{cl}}$ (both sub-laminations of $\mathcal{L}^{\mathrm{fu}}$) with transversal $X^{\bullet} = K_\rho \backslash X$ and two cluster equivalence relations $\mathcal{R}_1^{\mathrm{cl}}$ and $\mathcal{R}_2^{\mathrm{cl}}$ (both subrelations of $\mathcal{R}^{\mathrm{fu}}$).

Remark 4.1 *An invariant coupling always exists since the product space with the product measure and the diagonal action will do. In case the coupling witnesses a stochastic domination*[23] *(see for instance [HJL02a, sect. 2.2]), then $\mathcal{R}_1^{\mathrm{cl}}$ fits into $\mathcal{R}_2^{\mathrm{cl}}$: $\mathcal{R}_1^{\mathrm{cl}} \subset \mathcal{R}_2^{\mathrm{cl}}$.*

Theorem 4.2 *Let $\mathcal{G}$ be a unimodular transitive locally finite graph. Let H be a unimodular transitive group of automorphisms of $\mathcal{G}$, let (X, μ) be an H-equivariant coupling between two H-invariant percolations μ_1, μ_2. Assume that*

1. *μ_1-a.e. cluster belongs to $\mathcal{O}_{\mathbf{HD}}$,*
2. *π_2 has an H-equivariant selected cluster defined on a non-null set*[24]*,*

then

$$\beta_1(\mathcal{G}) \leq \frac{1}{2} \sum_{\substack{\text{edges } e \\ \text{adjacent to } \rho}} \mu\Big(\pi_2(e) = 1 \text{ and } \pi_1(e) = 0\Big).$$

Here, $\beta_1(\mathcal{G})$ is the invariant of the graph introduced in Definition 2.10. It is strictly positive if and only if $\mathcal{G}$ doesn't belong to $\mathcal{O}_{\mathbf{HD}}$.

The main ingredient in the proof of the theorem will be the following useful result. Here, graphing may be understood as oriented or unoriented (see Section 8).

Theorem 4.3 *Let $\mathcal{R}_1$ be a p.m.p. equivalence relation on the standard Borel space (X, μ). Let Ψ_2 be a p.m.p. graphing and let $\mathcal{R}_2 = \mathcal{R}_1 \vee \Psi_2$ be the equivalence relation generated by $\mathcal{R}_1$ and Ψ_2. Then*

$$\beta_1(\mathcal{R}_2) - \beta_0(\mathcal{R}_2) \leq \beta_1(\mathcal{R}_1) - \beta_0(\mathcal{R}_1) + \mathrm{cost}(\Psi_2).$$

Remark 4.4 *There is no continuity in the other direction: Think in $\mathcal{R}_i$ given by a free action of $\Gamma_1 = \mathbf{F}_2$ and $\Gamma_2 = \mathbf{F}_2 \times \mathbb{Z}$. Then $\mathcal{R}_2$ can be generated from $\mathcal{R}_1$ by adding a graphing Ψ_2 of arbitrarily small cost. However, $\beta_1(\mathcal{R}_2) - \beta_0(\mathcal{R}_2) = 0$ while $\beta_1(\mathcal{R}_1) - \beta_0(\mathcal{R}_1) = 1$.*

PROOF: The proof is just an adaptation of the proof of the Morse inequalities (see [Gab02, sect. 4.4, p.137]). Let $\bar{\Sigma}_1$ be a *simply connected* smooth simplicial $\mathcal{R}_1$-complex, with a big enough 0-skeleton: $\mathcal{R}_1 \subset \bar{\Sigma}_1$. Then $(\beta_1 - \beta_0)(\mathcal{R}_1) = (\beta_1 - \beta_0)(\bar{\Sigma}_1)$. Let $(\bar{\Sigma}_1^n)_n$ be an increasing sequence of ULB smooth simplicial $\mathcal{R}_1$-complexes that exhausts $\bar{\Sigma}_1$, with say $\mathcal{R}_1 \subset \bar{\Sigma}_1^0$.

[23] i.e. $\mu\{x : \pi_1(x) \leq \pi_2(x)\} = 1$, i.e. for μ a.e. $x \in X$ and for every edge $e \in \mathtt{E}$, if $\pi_1(x)(e) = 1$, then $\pi_2(x)(e) = 1$.

[24] For instance, if μ_2 has a non-null set of subgraphs with exactly one infinite cluster.

Let $\bar{\bar{\Sigma}}_1$ and $\bar{\bar{\Sigma}}_1^n$ be the corresponding smooth $\mathcal{R}_2$-complexes ([Gab02, sect. 5.2, p.140]): $\mathcal{R}_2 \subset \bar{\bar{\Sigma}}_1^0$ and moreover, the reciprocity formula gives

$$(\beta_1 - \beta_0)(\mathcal{R}_1) = \lim_n (\beta_1 - \beta_0)(\bar{\Sigma}_1^n, \mathcal{R}_1) = \lim_n (\beta_1 - \beta_0)(\bar{\bar{\Sigma}}_1^n, \mathcal{R}_2).$$

Define $\bar{\bar{\Sigma}}_2$ and $\bar{\bar{\Sigma}}_2^n$ by just adding to $\bar{\bar{\Sigma}}_1$ and $\bar{\bar{\Sigma}}_1^n$ the $\mathcal{R}_2$-field of graphs associated to Ψ_2 on $\mathcal{R}_2$.
Claim: $(\beta_1 - \beta_0)(\bar{\bar{\Sigma}}_2^n) \leq (\beta_1 - \beta_0)(\bar{\bar{\Sigma}}_1^n) + \mathrm{cost}(\Psi_2)$, $\forall n$.
This is an immediate computation by mimicking the first 4 lines of [Gab02, sect. 4.4, p.137] (read $b_0 = 0$ instead of $b_{-1} = 0$, there).
Now, by letting n tend to ∞, $(\beta_1 - \beta_0)(\bar{\bar{\Sigma}}_2) \leq (\beta_1 - \beta_0)(\bar{\bar{\Sigma}}_1) + \mathrm{cost}(\Psi_2) = (\beta_1 - \beta_0)(\mathcal{R}_1) + \mathrm{cost}(\Psi_2)$.
By definition, $\bar{\bar{\Sigma}}_2$ is connected, so that $\beta_0(\bar{\bar{\Sigma}}_2) = \beta_0(\mathcal{R}_2)$ [Gab02, 3.14]. On the other hand, $\beta_1(\mathcal{R}_2) \leq \beta_1(\bar{\bar{\Sigma}}_2)$ [Gab02, 3.13 and 3.14]. This proves Th. 4.3. ■

Proof (of Theorem 4.2): Denote by $\check{\mathcal{R}} = \mathcal{R}_1^{\mathrm{cl}} \vee \mathcal{R}_2^{\mathrm{cl}}$ the equivalence relation generated by $\mathcal{R}_1^{\mathrm{cl}}$ and $\mathcal{R}_2^{\mathrm{cl}}$. It is also the relation defined on $X^\bullet$ by the union lamination $\mathcal{L}_1^{\mathrm{cl}} \cup \mathcal{L}_2^{\mathrm{cl}}$.
Let $Y \subset X^\bullet$ be a $\mu^\bullet$-non-null $\check{\mathcal{R}}$-saturated Borel subset. It is clear from the lamination description that the restriction $\check{\mathcal{R}}_{|Y}$ is generated by the restriction $\mathcal{R}_{1|Y}^{\mathrm{cl}}$ together with the graphing Ψ_2 consisting of edges of $\mathcal{L}_2^{\mathrm{cl}} \setminus \mathcal{L}_1^{\mathrm{cl}}$ with both endpoints in Y (one endpoint in Y implies the other one in Y, by saturation):

$$\check{\mathcal{R}}_{|Y} = \mathcal{R}_{1|Y}^{\mathrm{cl}} \vee \Psi_2.$$

The cost of Ψ_2 (again, $\frac{1}{2}$ just reflects that each edge is counted twice, while $\frac{1}{\mu^\bullet(Y)}$ is just designed to normalize) is bounded above by

$$\mathrm{cost}(\Psi_2) \leq \frac{1}{2\mu^\bullet(Y)} \sum_{\substack{\text{edges } e \\ \text{adjacent to } \rho}} \mu\Big(\pi_2(e) = 1 \text{ and } \pi_1(e) = 0\Big).$$

The above Theorem 4.3 gives:

$$\beta_1(\check{\mathcal{R}}_{|Y}) - \beta_0(\check{\mathcal{R}}_{|Y}) \leq \beta_1(\mathcal{R}_{1|Y}^{\mathrm{cl}}) - \beta_0(\mathcal{R}_{1|Y}^{\mathrm{cl}}) + \frac{1}{2\mu^\bullet(Y)} \sum_{\substack{\text{edges } e \\ \text{adjacent to } \rho}} \mu\Big(\pi_2(e) = 1 \text{ and } \pi_1(e) = 0\Big).$$

If Z is a measurable non-null subset where ρ belongs to the H-equivariant selected cluster (for instance, the unique infinite cluster), then $\mathcal{R}_{2|Z}^{\mathrm{cl}} = \mathcal{R}_{|Z}^{\mathrm{fu}}$. So that its $\check{\mathcal{R}}$-saturation Y satisfies $\check{\mathcal{R}}_Y = \mathcal{R}_Y^{\mathrm{fu}}$ and thus (see fact 2, Subsection 1.4)

$$\mu^\bullet(Y)\beta_1(\check{\mathcal{R}}_{|Y}) = \mu^\bullet(Y)\beta_1(\mathcal{R}_{|Y}^{\mathrm{fu}}) = \beta_1(\mathcal{R}^{\mathrm{fu}}),$$

which coincides by definition with the quantity $\beta_1(\mathcal{G})$ introduced in Section 2.4, Definition 2.10.
Recall that $\beta_1(\mathcal{G}) = 0$ iff $\mathcal{G} \in \mathcal{O}_{\mathbf{HD}}$. The graph $\mathcal{G}$ being infinite, $\beta_0(\check{\mathcal{R}}_{|Y}) = \beta_0(\mathcal{R}_{|Y}^{\mathrm{fu}}) = 0$, so that

$$0 < \beta_1(\mathcal{G}) \leq \mu^\bullet(Y)\big(\beta_1(\mathcal{R}_{1|Y}^{\mathrm{cl}}) - \beta_0(\mathcal{R}_{1|Y}^{\mathrm{cl}})\big) + \frac{1}{2} \sum_{\substack{\text{edges } e \\ \text{adjacent to } \rho}} \mu\Big(\pi_2(e) = 1 \text{ and } \pi_1(e) = 0\Big).$$

Now, the assumption (1) of Theorem 4.2 is designed (since $\beta_1(\mathcal{R}^{\mathrm{cl}}_{1|Y}) = 0$ by Th. 0.3) to ensure that

$$\beta_1(\mathcal{R}^{\mathrm{cl}}_{1|Y}) - \beta_0(\mathcal{R}^{\mathrm{cl}}_{1|Y}) \leq 0$$

and this finishes its proof. ■

4.1 Application to Bernoulli Percolation

Corollary 4.5 *(Th. 0.6) Let $\mathcal{G}$ be a unimodular transitive locally finite graph. If $\mathcal{G}$ doesn't belong to $\mathcal{O}_{\mathbf{HD}}$, then the nonuniqueness phase interval of Bernoulli percolation has non-empty interior:*

$$p_c(\mathcal{G}) < p_u(\mathcal{G})$$

More precisely,

$$0 < \beta_1(\mathcal{G}) \leq \frac{1}{2}(\text{degree of } \mathcal{G})(p_u(\mathcal{G}) - p_c(\mathcal{G})).$$

Proof: The standard coupling $([0,1]^{\mathtt{E}}, \otimes\text{Lebesgue}) \xrightarrow{\pi_p} (\{0,1\}^{\mathtt{E}}, \mathbf{P}_p)$ (see for example Section 1.3.d) provides a family of countable equivalence relations $\mathcal{R}^{\mathrm{cl}}_p$ (on the quotient space $K_\rho \backslash [0,1]^{\mathtt{E}}$, once given a closed unimodular transitive group of automorphisms of $\mathcal{G}$). For $s < p_c$, the equivalence classes of the cluster equivalence relation are a.s. finite, thus μ_s-a.e. cluster belongs to $\mathcal{O}_{\mathbf{HD}}$. The right-hand quantity of Theorem 4.2 with $\mu_1 = \mu_s$ and $\mu_2 = \mu_t$, for t in the uniqueness phase, is

$$\frac{1}{2} \sum_{\substack{\text{edges } e \\ \text{adjacent to } \rho}} \mu\Big(\pi_2(e) = 1 \text{ and } \pi_1(e) = 0\Big) \;=\; \frac{1}{2}(\text{degree of } \mathcal{G})(t - s).$$

One concludes by continuity, by letting s tend to $p_c(\mathcal{G})$ and t tend to $p_u(\mathcal{G})$.

Observe that one could have applied Theorem 4.2 directly with $s = p_c(\mathcal{G})$: there is almost surely no infinite cluster at $p_c(\mathcal{G})$ [BLPS99a, Th. 1.3]. ■

Remark 4.6 While the above corollary extends to unimodular quasi-transitive locally finite graphs, it is unknown whether the unimodularity assumption may be removed. On the other hand, the removal of any transitivity assumption makes it false since R. Lyons and Y. Peres showed (personal communication: I want to thank them allowing me to reproduce their description here) that the following graph $\mathcal{G}$ doesn't belong to $\mathcal{O}_{\mathbf{HD}}$ but on the other hand, the set of parameters p in Bernoulli percolation doesn't admit any interval of nonuniqueness.

Denote by $\mathcal{G}_m$ the graph obtained from the lattice $\mathbb{Z}^2$ by replacing each edge of $\mathbb{Z}^2$ by m paths of length 2. We fix m large enough so that $p_c(\mathcal{G}_m) < p_c(\mathbb{Z}^3)$. Denote by $\mathcal{G}_m(k)$ a k-by-k square in $\mathbb{Z}^2$, with each edge replaced by m paths of length 2. Now consider two copies of $\mathbb{Z}^3$ (call them $\mathcal{G}'$ and $\mathcal{G}''$) that we will join in countably many corresponding places $(x_i, y_i) \in \mathcal{G}' \times \mathcal{G}''$, with density 0 in both $\mathcal{G}'$ and $\mathcal{G}''$, using graphs $\mathcal{G}_m(k_i)$ as follows: position $\mathcal{G}_m(k_i)$ with one corner at x_i and another corner at y_i. We make k_i grow fast enough so that the effective conductance between $\mathcal{G}'$ and $\mathcal{G}''$ is finite; explicitly, we make $\sum_i 1/(\log k_i) < \infty$. This constructs the graph $\mathcal{G}$.

4.2 Application to Random-Cluster Model

Corollary 4.7 *(Th. 0.7) Let $\mathcal{G}$ be a unimodular transitive locally finite graph, not in $\mathcal{O}_{\mathbf{HD}}$. Fix the parameter $q \in [1, \infty)$. The gap between the left limit (when $p \nearrow p_c(q)$) and the right limit (when $p \searrow p_u(q)$) of the expected degree of a base point ρ with respect to the measure $\mathtt{RC}_{p,q}$ satisfies:*

$$0 < \beta_1(\mathcal{G}) \leq \frac{1}{2}\Big(\mathtt{RC}_{p_u+,q}[\deg(\rho)] - \mathtt{RC}_{p_c-,q}[\deg(\rho)]\Big).$$

Here, $\mathtt{RC}$ denotes either $\mathtt{WRC}$ or $\mathtt{FRC}$.

Proof: Consider the invariant coupling introduced by O. Häggström, J. Jonasson and R. Lyons in [HJL02a] of (all) the measures $\mathtt{FRC}_{p,q}$ and $\mathtt{WRC}_{p,q}$ (together) for $p \in [0,1]$ and $q \in [1, \infty)$

$$(X, \mu) \overset{\pi^{\mathtt{RC}}_{p,q}}{\to} (\{0,1\}^{\mathtt{E}}, \mathtt{RC}_{p,q})$$

It provides two families of countable equivalence relations $\mathcal{R}^{\mathrm{cl}}_{p,q}$ (on the quotient space $K_\rho \backslash X$, once given a closed unimodular transitive group of automorphisms of $\mathcal{G}$), one for $\mathtt{FRC}$ and one for $\mathtt{WRC}$. The usefulness of that coupling is that it reflects the stochastic domination (see [HJL02a, sect. 3]); in particular, for a fixed parameter q and $s < t$ (and denoting $\pi^{\mathtt{RC}}_{s,q}$ by π_s):

$$\mu\Big(\pi_t(e) = 1 \text{ and } \pi_s(e) = 0\Big) \quad = \quad \mu(\pi_t(e) = 1) - \mu(\pi_s(e) = 1)$$

Take s, t such that $s < p_c(q) \leq p_u(q) < t$, then Theorem 4.2 says that:

$$\beta_1(\mathcal{G}) \leq \frac{1}{2} \sum_{\substack{\text{edges } e \\ \text{adjacent to } \rho}} \mu(\pi_t(e) = 1) - \mu(\pi_s(e) = 1)$$

Now, the right member is precisely: $\frac{1}{2}\big(\mathtt{RC}_{t,q}[\deg(\rho)] - \mathtt{RC}_{s,q}[\deg(\rho)]\big)$. The monotonicity properties of the measures $\mathtt{RC}$ lead to the required inequality. Indeed, monotonicity as well as left continuity of $p \mapsto \mathtt{FRC}_{p,q}[\deg(\rho)]$ and right continuity of $p \mapsto \mathtt{WRC}_{p,q}[\deg(\rho)]$ follow, like in [HJL02b], from the fact that $\mathtt{FRC}$ is an increasing (and $\mathtt{WRC}$ is a decreasing) limit of increasing (in p) continuous functions. ■

5 Quasi-transitive graphs

This section indicates how to extend the above results to the context of quasi-transitive graphs, instead of just transitive ones. There is no qualitative reversal, and just some quantitative adjustments. The proofs are straightforward adaptions of those of the transitive case with just slight changes of notation. We first describe how to modify section 2.

Let $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ be a locally finite quasi-transitive graph, H a closed subgroup of $\mathrm{Aut}(\mathcal{G})$ whose action on $\mathtt{V}$ has finitely many orbits. Choose one vertex $\rho_1, \rho_2, \cdots, \rho_q$ in each orbit and denote by $K_1 = K_{\rho_1}, K_2 = K_{\rho_2}, \cdots, K_q = K_{\rho_q}$ its stabilizer.

Let (X, μ) be a standard Borel probability space together with

- a probability measure-preserving (p.m.p.) action of H, which is *essentially free*, and
- an H-equivariant Borel map $\pi : X \to \{0,1\}^{\mathtt{E}}$.

Divide out $X \times \mathcal{G}$ by the diagonal action of H to get the laminated space $\mathcal{L}^{\mathrm{fu}} = H \backslash (X \times \mathcal{G})$: the **full lamination**. Corresponding to the partition of $\mathtt{V}$ into H-orbits $\mathtt{V}_1, \mathtt{V}_2, \cdots, \mathtt{V}_q$, the transversal $X^\bullet :=$

$H\backslash(X \times \mathtt{V})$ identifies with the disjoint union of standard Borel spaces $K_1\backslash X \coprod K_2\backslash X \coprod \cdots \coprod K_q\backslash X$ via $X \times \mathtt{V}_i \ni (hx, h\gamma\rho_i) \mapsto K_i\gamma^{-1}x \in K_i\backslash X$. The leaf of $\mu^\bullet$-almost every $x^\bullet \in X^\bullet$ is isomorphic to $\mathcal{G}$.

Define the **full equivalence relation** $\mathcal{R}^{\mathrm{fu}}$ on $X^\bullet$ by $x^\bullet \mathcal{R}^{\mathrm{fu}} y^\bullet$ if and only if $x^\bullet$ and $y^\bullet$ are vertices of the same $\mathcal{L}^{\mathrm{fu}}$-leaf.

It inherits naturally an *unoriented graphing* and a *smooth field of graphs* (see Section 8, and examples 7.1, 7.3 of Section 7) from the edge set $H\backslash(X \times \mathtt{E})$, where the graph associated with each point admits an isomorphism with $\mathcal{G}$.

Consider, on $X^\bullet$, the probability measure $\mu^\bullet := \frac{1}{T}(\frac{\Pi_{1*}\mu}{m(K_1)} + \frac{\Pi_{2*}\mu}{m(K_2)} + \cdots + \frac{\Pi_{p*}\mu}{m(K_q)})$, where $T = \frac{1}{m(K_1)} + \frac{1}{m(K_2)} + \cdots + \frac{1}{m(K_q)}$ and $\Pi_{i*}\mu$ is the pushed-forward measure by $\Pi_i : X \to K_i\backslash X$. It is preserved by the equivalence relation $\mathcal{R}^{\mathrm{fu}}$ if and only if the group H is unimodular (Th. 2.1). Observe that with this choice (under the unimodularity asumption), the description depends neither on the choice of scaling of the Haar measure, nor on the choice of a particular orbit of vertices. The **first L^2 Betti number** of the equivalence relation $\mathcal{R}^{\mathrm{fu}}$ is given by the formula:

$$\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu^\bullet) := \frac{1}{2T}\sum_{i=1}^{q} \frac{1}{m(K_i)} \sum_{j=1}^{n_i} \langle p(\mathbf{1}_{e_{i,j}}) | \mathbf{1}_{e_{i,j}} \rangle,$$

where p is the projection $p : C^1_{(2)}(\mathcal{G}) \to d\mathbf{HD}(\mathcal{G}) \simeq \mathbf{HD}(\mathcal{G})/\mathbb{C}$, and for each $i = 1, 2, \cdots, q$, the vectors $\mathbf{1}_{e_{i,1}}, \mathbf{1}_{e_{i,2}}, \cdots, \mathbf{1}_{e_{i,n_i}} \in C^1_{(2)}(\mathcal{G})$ are the characteristic functions of (all) the edges $e_{i,1}, e_{i,2}, \cdots, e_{i,n_i}$ adjacent to the orbit representative ρ_i (see sect. 2.4). It is not hard to check that this quantity doesn't depend on the particular unimodular quasi-transitive group of automorphisms H[25 26]. One defines the **first ℓ^2 Betti number** of $\mathcal{G}$ by the same formula (see def. 2.10):

$$\beta_1(\mathcal{G}) := \frac{1}{2T}\sum_{i=1}^{q} \frac{1}{m(K_i)} \sum_{j=1}^{n_i} \langle p(\mathbf{1}_{e_{i,j}}) | \mathbf{1}_{e_{i,j}} \rangle.$$

Clearly, $\beta_1(\mathcal{G}) = 0$ if and only if $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$ (see footnote 21).

Example 5.1 *The leading example is the group $H = \mathbb{Z}/r\mathbb{Z} * \mathbb{Z}/s\mathbb{Z}$ acting on its Bass-Serre tree $\mathcal{G}$: the bipartite tree with valencies r and s. The stabilizers are $K_1 = \mathbb{Z}/r\mathbb{Z}$ and $K_2 = \mathbb{Z}/s\mathbb{Z}$ and the components of $X^\bullet \simeq (\mathbb{Z}/r\mathbb{Z})\backslash X \coprod (\mathbb{Z}/s\mathbb{Z})\backslash X$ identifie with pieces of X of measure $\frac{1}{r}$ and $\frac{1}{s}$. By considering the cost of the graphing inherited by $\mathcal{R}^{\mathrm{fu}}$ (see example 7.3 of Section 7 and Section 8), one computes: $\beta_1(\mathcal{G}) = \beta_1(\mathcal{R}^{\mathrm{fu}}, \mu^\bullet) = (\frac{1}{r} + \frac{1}{s})^{-1}[1 - (\frac{1}{r} + \frac{1}{s})]$. In the particular case of the regular tree of even valency $r = s = 2t$, $\beta_1(\mathcal{R}^{\mathrm{fu}}, \mu^\bullet) = t - 1 = \beta_1(\mathbf{F}_t)$ coincide for $H = \mathbb{Z}/2t\mathbb{Z} * \mathbb{Z}/2t\mathbb{Z}$ or $H = \mathbf{F}_t$.*

Thanks to the map $\pi : X \to \{0, 1\}^{\mathtt{E}}$, each leaf of $\mathcal{L}^{\mathrm{fu}}$ becomes a colored graph. The **cluster lamination** is obtained by removing all the 0-colored edges. Define the **cluster equivalence relation** $\mathcal{R}^{\mathrm{cl}}$ on $X^\bullet$ by $x^\bullet \mathcal{R}^{\mathrm{cl}} y^\bullet$ if and only if $x^\bullet$ and $y^\bullet$ are vertices of the same $\mathcal{L}^{\mathrm{cl}}$-leaf. It is a subrelation of $\mathcal{R}^{\mathrm{fu}}$.

The proof of Theorem 3.9 extends to quasi-transitive graphs with no modification.

[25] this is in fact true for the convex combination $\frac{1}{T}\sum_{i=1}^{q} \frac{1}{m(K_i)} \sum_{j=1}^{n_i} \eta(e_{i,j})$ associated with any $\mathrm{Aut}(\mathcal{G})$-invariant function η defined on the edges of $\mathcal{G}$.

[26] It is most probably the case that the higher dimensional L^2 Betti numbers $\beta_n(\mathcal{R}^{\mathrm{fu}}, \mu^\bullet)$ are also invariants of the graph (in fact, of the automorphism group of the unimodular graph, with a normalization given by an appropriate combination of the Haar measures of the stabilizers of the vertices), but this would move us apart from the purpose of this paper.

Theorem 3.9* *Assume that the closed subgroup H is unimodular and $\pi_*\mu$ admits an H-equivariant virtually selected cluster. If $\mathcal{G}$ belongs to $\mathcal{O}_{\mathbf{HD}}$, then μ-a.e. virtually selected cluster belongs to $\mathcal{O}_{\mathbf{HD}}$. If $\mathcal{G}$ doesn't belong to $\mathcal{O}_{\mathbf{HD}}$, then μ-a.e. virtually selected cluster doesn't belong to $\mathcal{O}_{\mathbf{HD}}$.*
In particular, on the Borel set of subgraphs with one infinite cluster, the infinite cluster belongs (resp. doesn't belong) to $\mathcal{O}_{\mathbf{HD}}$ a.s. iff $\mathcal{G}$ belongs (resp. doesn't belong) to $\mathcal{O}_{\mathbf{HD}}$.

We now turn to give the modifications in the statements of the quantitative estimates of section 4.

Theorem 4.2* *Consider a unimodular quasi-transitive group H of automorphisms of $\mathcal{G}$, two H-invariant percolations μ_1 and μ_2 on $\mathcal{G}$ and an H-**equivariant coupling***

$$\begin{array}{ccc} & (X,\mu) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (\{0,1\}^{\mathtt{E}},\mu_1) & & (\{0,1\}^{\mathtt{E}},\mu_2) \end{array}$$

Assume that
1. μ_1-a.e. cluster belongs to $\mathcal{O}_{\mathbf{HD}}$,
2. π_2 has an H-equivariant selected cluster defined on a non-null set[27],
then

$$\beta_1(\mathcal{G}) \leq \frac{1}{2T}\Big[\sum_{i=1}^{q}\frac{1}{m(K_i)}\sum_{\substack{\text{edges } e \\ \text{adjacent to } \rho_i}}\mu\Big(\pi_2(e)=1 \text{ and } \pi_1(e)=0\Big)\Big].$$

The only modifications in the proof of Theorem 4.2 are the bound on $\mathrm{cost}(\Psi_2)$:

$$\mathrm{cost}(\Psi_2) \leq \frac{1}{2\mu^\bullet(Y)}\frac{1}{T}\sum_{i=1}^{q}\frac{1}{m(K_i)}\sum_{\substack{\text{edges } e_{i,j} \\ \text{adjacent to } \rho_i}}\mu\Big(\pi_2(e)=1 \text{ and } \pi_1(e)=0\Big),$$

and the definition of the measurable subset $Z \subset X^\bullet$: For at least one of the ρ_i, the set of those $x \in X$ whose selected cluster contains ρ_i is non-null. Take for Z its image in $X^\bullet$. ∎

Application to Bernoulli Percolation (subsection 4.1*).

Corollary 4.5* *Let $\mathcal{G}$ be a unimodular quasi-transitive locally finite graph. If $\mathcal{G}$ doesn't belong to $\mathcal{O}_{\mathbf{HD}}$, then the nonuniqueness phase interval of Bernoulli percolation has non-empty interior:*

$$p_c(\mathcal{G}) < p_u(\mathcal{G})$$

More precisely,

$$0 < \beta_1(\mathcal{G}) \leq \frac{1}{2T}\sum_{i=1}^{q}\frac{\deg(\rho_i)}{m(K_i)}(p_u(\mathcal{G}) - p_c(\mathcal{G})).$$

where $\deg(\rho_i)$ is the number of edges in $\mathcal{G}$ that are adjacent to ρ_i. ∎

[27] For instance, if μ_2 has a non-null set of subgraphs with exactly one infinite cluster.

Application to Random-Cluster Model (subsection 4.2*).

Corollary 4.7* *Let $\mathcal{G}$ be a unimodular quasi-transitive locally finite graph, not in $\mathcal{O}_{\mathbf{HD}}$. Fix the parameter $q \in [1, \infty)$. The gap between the left limit (when $p \nearrow p_c(q)$) and the right limit (when $p \searrow p_u(q)$) of the expected degree of a base point ρ with respect to the measure $\mathtt{RC}_{p,q}$ satisfies:*

$$0 < \beta_1(\mathcal{G}) \leq \frac{1}{2T} \sum_{i=1}^{q} \frac{1}{m(K_i)} \Big(\mathtt{RC}_{p_u+,q}[\deg(\rho_i)] - \mathtt{RC}_{p_c-,q}[\deg(\rho_i)] \Big).$$

Here, $\mathtt{RC}$ either denotes $\mathtt{WRC}$ or $\mathtt{FRC}$, and $\mathtt{RC}_{p,q}[\deg(\rho_i)]$ is the mean degree of the vertex ρ_i in the random subgraph for the random-cluster measure $\mathtt{RC}$ with parameters p and q. Also $\mathtt{RC}_{p_u+,q}[\deg(\rho_i)] := \lim_{p \searrow p_u(q)} \mathtt{RC}_{p,q}[\deg(\rho_i)]$ and $\mathtt{RC}_{p_c-,q}[\deg(\rho_i)] := \lim_{p \nearrow p_c(q)} \mathtt{RC}_{p,q}[\deg(\rho_i)]$. ∎

6 Harmonic Dirichlet Functions and ℓ^2 Cohomology

The main result of the section is the technical Proposition 6.6 that will allow to make the connection between harmonic Dirichlet functions and the definitions of L^2 Betti numbers for equivalence relations in [Gab02] (see the proof of Theorem 7.5). However, as a leading and motivating example, we will consider the following well-known result relating harmonic Dirichlet functions with the first ℓ^2 Betti number.

Theorem 6.1 *Let Γ be a finitely generated group. Its first ℓ^2 Betti number $\beta_1(\Gamma)$ is not zero if and only if any of its Cayley graphs admits nonconstant harmonic Dirichlet functions.*

6.1 Harmonic Dirichlet Functions...

Let $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ be a connected graph [28] with bounded degree. The tail and head of an oriented edge $\hat{e}$ are denoted by $\hat{e}^-$ and $\hat{e}^+$.

Denote by $\mathcal{F}(\mathtt{V})$ or $C^0(\mathcal{G})$ the space of all complex-valued functions (0-cochains) on $\mathtt{V}$ and by $C^1(\mathcal{G})$ the space of 1-cochains [29]. Define the coboundary, "boundary" and Laplace maps:

$$\begin{array}{rcclrl}
d: & C^0(\mathcal{G}) & \longrightarrow & C^1(\mathcal{G}) & df(\hat{e}) = & f(\hat{e}^+) - f(\hat{e}^-) \\
d^*: & C^1(\mathcal{G}) & \longrightarrow & C^0(\mathcal{G}) & d^*g(v) = & \displaystyle\sum_{\{\hat{e}:\, \hat{e}^+=v\}} g(\hat{e}) \\
\Delta: & C^0(\mathcal{G}) & \longrightarrow & C^0(\mathcal{G}) & \Delta f(v) = d^*df(v) = & \displaystyle\sum_{\{\hat{e}:\, \hat{e}^+=v\}} (f(\hat{e}^+) - f(\hat{e}^-)) \\
 & & & & = & \deg(v) f(v) - \displaystyle\sum_{\{\hat{e}:\, \hat{e}^+=v\}} f(\hat{e}^-)
\end{array}$$

The spaces of ℓ^2-cochains will be denoted by $C^0_{(2)}(\mathcal{G})$ and $C^1_{(2)}(\mathcal{G})$. By definition, the space of **harmonic Dirichlet functions** on $\mathcal{G}$ is the space of functions whose value at each v equals the mean of the values at its neighbors ($\Delta(f) = 0$, i.e. f is harmonic) and with coboundary in ℓ^2 (f has finite energy or finite Dirichlet sum):

$$\mathbf{HD}(\mathcal{G}) := \{f \in \mathcal{F}(\mathtt{V}) : df \in C^1_{(2)}(\mathcal{G}) \text{ and } \Delta f = 0\}.$$

[28] In the whole Section 6, except in the examples, $\mathcal{G}$ is not assumed to have any kind of symmetries or automorphisms.

[29] i.e. the space of anti-symmetric functions on the set of oriented edges: $f(\check{e}) = -f(\hat{e})$ where $\check{e}$ is the edge $\hat{e}$ with the reverse orientation

The kernel of d clearly consists of the constant functions (since $\mathcal{G}$ is connected), so that naturally

$$\mathbf{HD}(\mathcal{G})/\mathbb{C} \;\simeq\; d(\mathbf{HD}(\mathcal{G})) = \mathrm{Im}d \cap C^1_{(2)}(\mathcal{G}) \cap \mathrm{Ker}d^*. \tag{4}$$

As for the ℓ^2 cohomology, it is not really that of the graph $\mathcal{G}$ that is of interest, since, for example, for Cayley graphs it is too sensitive to changes of generators (think of Cayley graphs of $\mathbb{Z}$, where $\bar{H}^1_{(2)} = 0$ or $\neq 0$ according to whether the generating system is (1) or $(2,3)$). One has first to "fill in the holes" of the graph:
Consider a simply-connected 2-dimensional complex Σ, with $\mathcal{G}$ as 1-skeleton.

Example 6.2 *For instance, let Γ be a group given by a presentation with g generators and r relators. Recall that the **Cayley complex** of the presentation is a 2-dimensional complex built from a bouquet of g oriented circles labeled by the generators, together with r oriented disks labeled by the relators glued along their boundary to this 1-dimensional skeleton by following successively the circles associated with the labeling relator. Its fundamental group is (isomorphic to) Γ. The universal cover Σ of the Cayley complex is a (simply connected) 2-dimensional complex (with a free action of Γ and) with the Cayley graph of Γ as 1-skeleton.*

More generally, Σ can be obtained from $\mathcal{G}$ by gluing one oriented disk (thought of as a polygon) along its boundary to each circuit (and the opposite orientation for the reverse circuit).

Denote by $C^2(\Sigma)$ the space of 2-cochains, i.e. anti-symmetric functions on the oriented 2-cells –disks–), by $C^2_{(2)}(\Sigma)$ the space of those that are ℓ^2 (i.e. $\sum_\sigma h(\sigma)^2 < \infty$, where the sum is over all the 2-cells σ). The boundary of a 2-cell σ in Σ being a 1-cycle $D^*\sigma$, one defines the *coboundary* D by $Dg(\sigma) = g(D^*\sigma)$.

$$C^0(\Sigma) \;\xrightarrow{d}\; C^1(\Sigma) \;\xrightarrow{D}\; C^2(\Sigma)$$

By taking the adjoint, $C^1_{(2)}(\Sigma) \cap \mathrm{Ker}d^* = \left[dC^0_{(2)}(\Sigma)\right]^{\perp C^1_{(2)}}$, the orthogonal of $\mathrm{Im}d_{(2)} = dC^0_{(2)}(\Sigma)$ in $C^1_{(2)}(\Sigma)$. Since Σ is simply connected, $\mathrm{Im}d = \mathrm{Ker}D$. From formula (4) we get the natural isomorphisms

$$\mathbf{HD}(\mathcal{G})/\mathbb{C} \;\simeq\; \mathrm{Ker}D \cap C^1_{(2)}(\Sigma) \cap \left[dC^0_{(2)}(\Sigma)\right]^{\perp} \simeq \frac{\mathrm{Ker}D \cap C^1_{(2)}(\Sigma)}{\overline{\mathrm{Im}}d_{(2)}}, \tag{5}$$

where $\overline{\mathrm{Im}}d_{(2)}$ is the closure of the space $\mathrm{Im}d_{(2)}$.

6.2 ... and ℓ^2 cohomology

To define ℓ^2 cohomology of Σ, one is led to consider ℓ^2 cochains and restrictions of the coboundary maps:

$$C^0_{(2)}(\Sigma) \xrightarrow{d_{(2)}} C^1_{(2)}(\Sigma) \xrightarrow{D_{(2)}} C_{(2)}(\Sigma)$$

Say that Σ is **uniformly locally bounded** (ULB) if it admits a uniform bound M s.t. each vertex (resp. edge) belongs to at most M edges (resp. 2-cells), and the boundary of each 2-cell has length at most M. In this situation, $D_{(2)}$ is a bounded operator, and the standard **first reduced ℓ^2 cohomology space** of Σ is defined as the Hilbert space

$$\bar{H}^1_{(2)}(\Sigma) := \frac{\mathrm{Ker}D_{(2)}}{\overline{\mathrm{Im}}d_{(2)}} = \frac{\mathrm{Ker}D \cap C^1_{(2)}(\Sigma)}{\overline{\mathrm{Im}}d_{(2)}}.$$

It follows from (5) that for a ULB Σ

$$\mathbf{HD}(\mathcal{G})/\mathbb{C} \;\simeq \bar{H}^1_{(2)}(\Sigma). \tag{6}$$

Example 6.3 *Σ is ULB if it comes from the Cayley complex of a finitely presented group. One can show that any other simply connected,* cocompact *free Γ-complex Σ' leads to a (Γ-equivariantly) isomorphic $\bar{H}^1_{(2)}(\Sigma')$, so that the non-triviality of $\bar{H}^1_{(2)}(\Sigma)$ is an invariant of the group Γ, and not only of the complex Σ. The first ℓ^2 Betti number $\beta_1(\Gamma)$ is the von Neumann Γ-dimension $\dim_\Gamma \bar{H}^1_{(2)}(\Sigma)$ and we retain from this dimension theory that it vanishes iff $\bar{H}^1_{(2)}(\Sigma) = \{0\}$.*

Observe that some finiteness condition on Σ is however necessary since the free group $\mathbf{F}_2$, for example, acts on its Cayley tree as well as on the product of the tree with a line, whose 1-skeleton admits (resp. doesn't admit) nonconstant harmonic Dirichlet functions. In case the presentation of Γ is not finite ($r = \infty$), Σ is no longer Γ-cocompact, and $D_{(2)}$ is no longer a continuous (=bounded) map.

The general way to proceed to define ℓ^2 cohomology for a complex that is not ULB, in the spirit of J. Cheeger and M. Gromov [CG86], consists in approximating Σ by its ULB subcomplexes. Consider the directed set of ULB subcomplexes Σ_t of Σ, directed by inclusion and the inverse system of reduced ℓ^2 cohomology spaces $H^n_{(2)}(\Sigma_t)$ of Σ_t with the maps $H^n_{(2)}(\Sigma_s) \to H^n_{(2)}(\Sigma_t)$ induced by inclusion $\Sigma_s \supset \Sigma_t$ (denoted by $s \geq t$). Then define the reduced ℓ^2 cohomology as the inverse limit $H^n_{(2)}(\Sigma) := \varprojlim H^n_{(2)}(\Sigma_t)$.

In our context, all the ULB complexes Σ_t, as well as Σ itself, share the same ULB 1-skeleton $\mathcal{G}$. Thus, the first reduced ℓ^2 cohomology spaces $\bar{H}^1_{(2)}(\Sigma_t)$ are each the quotient of the subspace $\mathrm{Ker}\Big(C^1_{(2)}(\mathcal{G}) \xrightarrow{D} C^2_{(2)}(\Sigma_t)\Big)$ of $C^1_{(2)}(\mathcal{G})$ (organized into an inverse system by inclusion), by the common subspace $\overline{\mathrm{Im}} d_{(2)}$. It follows that:

$$\bar{H}^1_{(2)}(\Sigma) = \varprojlim \bar{H}^1_{(2)}(\Sigma_t) = \frac{\bigcap_t \mathrm{Ker}(C^1_{(2)}(\mathcal{G}) \xrightarrow{D} C^2_{(2)}(\Sigma_t))}{\overline{\mathrm{Im}} d_{(2)}} = \frac{\mathrm{Ker} D \cap C^1_{(2)}(\mathcal{G})}{\overline{\mathrm{Im}} d_{(2)}} \tag{7}$$

and by formula (5), valid for any Σ:

$$\bar{H}^1_{(2)}(\Sigma) \;\simeq\; \mathbf{HD}(\mathcal{G})/\mathbb{C}. \tag{8}$$

Example 6.4 *Let Γ be finitely generated, but not necessarily finitely presented. For a Γ-complex Σ, the Σ_t are moreover required to be Γ-invariant (and cocompact) and the ℓ^2 Betti numbers of the Γ-action on Σ are defined by keeping track of the Γ-dimensions:*

$$\begin{aligned} \beta_n(\Sigma, \Gamma) &:= \sup_t \dim_\Gamma \overline{\mathrm{Im}}(\bar{H}^n_{(2)}(\Sigma) \to \bar{H}^n_{(2)}(\Sigma_t)) \\ &= \sup_t \dim_\Gamma \cap_{s \geq t} \overline{\mathrm{Im}}(\bar{H}^n_{(2)}(\Sigma_s) \to \bar{H}^n_{(2)}(\Sigma_t)) \end{aligned}$$

For simply connected Γ-complexes Σ, the value $\beta_1(\Sigma, \Gamma)$ doesn't depend on a particular choice of Σ, so that for Σ constructed from a Cayley complex of Γ,

$$\beta_1(\Gamma) = \beta_1(\Sigma, \Gamma) = \dim_\Gamma \mathbf{HD}(\mathcal{G})/\mathbb{C}$$

vanishes if and only if $\mathbf{HD}(\mathcal{G})/\mathbb{C} = \{0\}$. This proves Theorem 6.1.

Let's quote for further use the observation that the space of formula (7) may be obtained by considering a exhausting sequence instead of the whole inverse system:

Proposition 6.5 *Let $\mathcal{G}$ be a graph with finite degree, Σ a simply-connected 2-dimensional complex with 1-skeleton $\mathcal{G}$. If $(\Sigma_t)_{t\in\mathbb{N}}$ is an increasing and exhausting sequence of ULB subcomplexes of Σ, then for any fixed t*

$$\cap_{s\geq t}\mathrm{Im}(\bar{H}^1_{(2)}(\Sigma_s)\to\bar{H}^1_{(2)}(\Sigma_t))=\frac{\mathrm{Ker}D\cap C^1_{(2)}(\mathcal{G})}{\overline{\mathrm{Im}}d_{(2)}}$$

doesn't depend on t and is NATURALLY isomorphic with $\mathbf{HD}(\mathcal{G})/\mathbb{C}$.

The connection with the simplicial framework of [Gab02] is made by considering a double barycentric subdivision Σ^* of Σ, with the exhaustion Σ^*_t corresponding to the subdivision of Σ_t. Since for each t, $\bar{H}^1_{(2)}(\Sigma_t)$ and $\bar{H}^1_{(2)}(\Sigma^*)$ are naturally isomorphic, it follows that

Proposition 6.6 *For any fixed t, $\cap_{s\geq t}\mathrm{Im}(\bar{H}^1_{(2)}(\Sigma^*_s)\to\bar{H}^1_{(2)}(\Sigma^*_t))$ doesn't depend on t and is NATURALLY isomorphic with $d(\mathbf{HD}(\mathcal{G}))$ and $\mathbf{HD}(\mathcal{G})/\mathbb{C}$.*

7 Fields of Graphs, Harmonic Dirichlet Functions and L^2 Betti Numbers for Equivalence Relations

Let (X,μ) be a standard Borel space with a probability measure μ and $\mathcal{R}$ a measure-preserving Borel equivalence relation with countable classes.

Recall from [Gab02] that an **$\mathcal{R}$-equivariant field $x\mapsto\Sigma_x$ of simplicial complexes** is a measurable assignment to each $x\in X$ of a simplicial complex Σ_x, together with an "action" of $\mathcal{R}$, i.e. with the measurable data of a simplicial isomorphism, for every $(x,y)\in\mathcal{R}$, $\psi_{x,y}:\Sigma_y\to\Sigma_x$ such that $\psi_{x,y}\psi_{y,z}=\psi_{x,z}$ and $\psi_{z,z}=id_{\Sigma_z}$. It is **smooth** if the action on the vertices admits a Borel fundamental domain. It is **smooth uniformly locally bounded** if there is a uniform bound N on the degree of the 1-skeleton of the Σ_x, and there is a Borel fundamental domain that meets each Σ_x in at most N vertices.

Example 7.1 *Let $\mathcal{R}$ be a p.m.p. countable Borel equivalence relation on the probability standard Borel space (X,μ). An unoriented graphing Ψ over $\mathcal{R}$ (see Section 8) defines an $\mathcal{R}$-equivariant field of graphs $x\mapsto\Psi_x$ with vertex set $\mathcal{R}$ itself, which is smooth.*
- The vertex set of Ψ_x is the set $\{(x,y)\in\mathcal{R}\}$, i.e. the set of elements of $\mathcal{R}$ with first coordinate x.
- Two vertices (x,y) and (x,z) of Ψ_x are neighbors if and only if (y,z) belongs to Ψ, i.e. iff the second coordinates are neighbor for Ψ
- The left action of $\mathcal{R}$ on itself $(w,x).(x,y)=(w,y)$ and thus on the set of vertices induces a natural action on the field: $(w,x):\begin{pmatrix}\Psi_x & \longrightarrow & \Psi_w\\ [(x,y),(x,z)] & \mapsto & [(w,y),(w,z)]\end{pmatrix}$.
- The "diagonal" set $\{(x,x):x\in X\}$ of vertices forms a Borel fundamental domain.

This example contains as main applications the various equivariant fields of graphs (described below) relevant for percolation theory.

Example 7.2 *In the context of Section 1 for a Cayley graph $\mathcal{G}$,*

($\mathcal{R}^{\mathrm{fu}}$, $x \mapsto \mathcal{G}$) *The full lamination $\mathcal{L}^{\mathrm{fu}}$ (Section 1.1) defines an unoriented graphing over $\mathcal{R}^{\mathrm{fu}}$ (see Section 8). In the corresponding smooth $\mathcal{R}^{\mathrm{fu}}$-equivariant field $x \mapsto \mathcal{L}^{\mathrm{fu}}_x$, each $\mathcal{L}^{\mathrm{fu}}_x$ admits a canonical isomorphism with $\mathcal{G}$.*

($\mathcal{R}^{\mathrm{cl}}$, $x \mapsto \pi(x)(\rho)$) *The cluster lamination $\mathcal{L}^{\mathrm{cl}}$ defines an unoriented graphing over $\mathcal{R}^{\mathrm{cl}}$. In the corresponding smooth $\mathcal{R}^{\mathrm{cl}}$-equivariant field $x \mapsto \mathcal{L}^{\mathrm{cl}}_x$, each $\mathcal{L}^{\mathrm{fu}}_x$ is isomorphic to the cluster $\pi(x)(\rho)$ of ρ in $\pi(x)$.*

($\mathcal{R}^{\mathrm{fu}}$, $x \mapsto \pi(x)$) *The cluster lamination $\mathcal{L}^{\mathrm{cl}}$ defines also an unoriented graphing over $\mathcal{R}^{\mathrm{fu}}$ and in the corresponding field $x \mapsto \Psi_x$, each Ψ_x is isomorphic to the subgraph $\pi(x)$, which is non-connected in general.*

Example 7.3 *In the context of Section 2 for a transitive, locally finite graph $\mathcal{G}$,*

($\mathcal{R}^{\mathrm{fu}}$, $x^\bullet \mapsto \mathcal{G}$) *The full lamination $\mathcal{L}^{\mathrm{fu}}$ (Section 2.1) defines an unoriented graphing over $\mathcal{R}^{\mathrm{fu}}$. In the corresponding smooth $\mathcal{R}^{\mathrm{fu}}$-equivariant field $x^\bullet \mapsto \mathcal{L}^{\mathrm{fu}}_{x^\bullet}$ each $\mathcal{L}^{\mathrm{fu}}_{x^\bullet}$ admits a (non-canonical) isomorphism with $\mathcal{G}$. However, each representative $x \in X$ of $x^\bullet$ defines an isomorphism j_x : $\mathcal{L}^{\mathrm{fu}}_{x^\bullet} \simeq \mathcal{G}$ and for another one $y = kx$, $j_y = kj_x$, where $k \in K_\rho$ so that this isomorphism is canonical up to an element of K_ρ.*

($\mathcal{R}^{\mathrm{cl}}$, $x^\bullet \mapsto \pi(x)(\rho)$) *The cluster lamination $\mathcal{L}^{\mathrm{cl}}$ defines an unoriented graphing over $\mathcal{R}^{\mathrm{cl}}$. The corresponding field $x^\bullet \mapsto \mathcal{L}^{\mathrm{cl}}_{x^\bullet}$ assigning to $x^\bullet$ its leaf (graph) in the lamination $\mathcal{L}^{\mathrm{cl}}$ is a smooth uniformly locally bounded $\mathcal{R}^{\mathrm{cl}}$-equivariant field of connected graphs. Each $\mathcal{L}^{\mathrm{fu}}_{x^\bullet}$ is isomorphic to the cluster $\pi(x)(\rho)$ of ρ for a (any) representative $x \in X$ of $x^\bullet$. Two representatives give subgraphs of $\mathcal{G}$ that are isomorphic under an element of K_ρ. The graph $\mathcal{L}^{\mathrm{cl}}_{x^\bullet}$ belongs to $\mathcal{O}_{\mathbf{HD}}$ iff $\pi(x)(\rho)$ belongs to $\mathcal{O}_{\mathbf{HD}}$ for any representative x of $x^\bullet$.*

Example 7.4 *(Restrictions) If $x \mapsto \Psi_x$ is a smooth $\mathcal{R}$-equivariant field of graphs and Y is a Borel subset then restricted to Y, the fields $Y \ni x \mapsto \Psi_x$ is a smooth $\mathcal{R}_{|Y}$-equivariant field of graphs, where $\mathcal{R}_{|Y}$ is the restriction of $\mathcal{R}$ to Y.*

Also recall from [Gab02] that there is a well-defined notion of L^2 Betti numbers $\beta_n(\mathcal{R}, \mu)$ for a measure-preserving Borel equivalence relation $\mathcal{R}$ with countable classes, which uses the notion of equivariant fields of simplicial complexes and the *von Neumann dimension* $\dim_{\mathcal{R}}$ associated with the von Neumann algebra of the equivalence relation and the measure μ.

Theorem 7.5 *Let $\mathcal{R}$ be a measure-preserving equivalence relation with countable classes on the standard Borel probability measure space (X, μ). Consider a smooth uniformly locally bounded $\mathcal{R}$-equivariant field $x \mapsto \mathcal{G}_x$ of connected graphs. Then*

$$\beta_1(\mathcal{R}, \mu) = \dim_{\mathcal{R}} \int_X^{\oplus} d(\mathbf{HD}(\mathcal{G}_x))\ d\mu(x) = \dim_{\mathcal{R}} \int_X^{\oplus} \mathbf{HD}(\mathcal{G}_x)/\mathbb{C}\ d\mu(x).$$

Since $\dim_{\mathcal{R}} H = 0$ if and only if $H = \{0\}$, one gets:

Corollary 7.6 *For a smooth uniformly locally bounded $\mathcal{R}$-equivariant field $x \mapsto \mathcal{G}_x$ of connected graphs, $\beta_1(\mathcal{R}, \mu) = 0$ if and only if μ a.s. $\mathcal{G}_x \in \mathcal{O}_{\mathbf{HD}}$.*

Remark 7.7 *Since a generating (oriented) graphing in the sense of [Lev95, Gab00] (see also Section 8) defines an unoriented graphing and thus a smooth $\mathcal{R}$-equivariant field of connected graphs, the above Corollary 7.6 reduces to Theorem 0.3 of the introduction.*

Proof: By Theorem/Definition [Gab02, Th. 3.13, Déf. 3.14], $\beta_1(\mathcal{R},\mu)$ is the first L^2 Betti number $\beta_1(\Sigma,\mathcal{R},\mu)$ of ANY smooth $\mathcal{R}$-equivariant field of *simply connected* (2-dimensional, say) simplicial complexes Σ. It can be computed [Gab02, prop. 3.9] by using any exhausting increasing sequence $(\Sigma_s)_{s\in\mathbb{N}}$ of $\mathcal{R}$-invariant uniformly locally bounded (ULB) sub-complexes by the following formula (9):

$$\begin{aligned}
\beta_1(\Sigma,\mathcal{R},\mu) &= \lim_{s\to\infty}\nearrow \lim_{s\geq t\ s\to\infty}\searrow \dim_{\mathcal{R}} \overline{\mathrm{Im}}[\bar{H}_1^{(2)}(\Sigma_t,\mathcal{R},\mu)\to\bar{H}_1^{(2)}(\Sigma_s,\mathcal{R},\mu)] &(9)\\
&= \lim_{t\to\infty}\nearrow \lim_{s\geq t\ s\to\infty}\searrow \dim_{\mathcal{R}} \overline{\mathrm{Im}}[\bar{H}^1_{(2)}(\Sigma_s,\mathcal{R},\mu)\to\bar{H}^1_{(2)}(\Sigma_t,\mathcal{R},\mu)] &(10)\\
&= \lim_{t\to\infty}\nearrow \dim_{\mathcal{R}} \bigcap_{s\geq t\ s\to\infty} \overline{\mathrm{Im}}[\bar{H}^1_{(2)}(\Sigma_s,\mathcal{R},\mu)\to\bar{H}^1_{(2)}(\Sigma_t,\mathcal{R},\mu)] &(11)
\end{aligned}$$

The equality (10) holds by duality between homology and cohomology because, just as in usual linear algebra, taking dual does not alter the dimension of the image. Equality (11) is due to the continuity of dimension, since $\overline{\mathrm{Im}}[\bar{H}^1_{(2)}(\Sigma_s,\mathcal{R},\mu)\to\bar{H}^1_{(2)}(\Sigma_t,\mathcal{R},\mu)]\subset\bar{H}^1_{(2)}(\Sigma_t,\mathcal{R},\mu)$ decreases with s. Now, for a fixed t, one has the Hilbert integral decomposition:

$$\bigcap_{s\geq t\ s\to\infty} \overline{\mathrm{Im}}[\bar{H}^1_{(2)}(\Sigma_s,\mathcal{R},\mu)\to\bar{H}^1_{(2)}(\Sigma_t,\mathcal{R},\mu)] = \int_X^{\oplus} \bigcap_{s\geq t\ s\to\infty} \overline{\mathrm{Im}}[\bar{H}^1_{(2)}(\Sigma_{s,x})\to\bar{H}^1_{(2)}(\Sigma_{t,x})]\,d\mu(x) \qquad (12)$$

It remains to make the choice of a Σ and of the sequence $(\Sigma_t)_{t\in\mathbb{N}}$ and to relate this with harmonic Dirichlet functions via Section 6. The simplicial complex $\Sigma_{t,x}$ is obtained from $\mathcal{G}_x$ by first gluing a disk along each circuit of length t and then taking the second barycentric subdivision. The simplicial complex Σ_x is their union. For each s, it follows from naturality in Proposition 6.6, applied for each x, that there is an isomorphism of Hilbert $\mathcal{R}$-modules:

$$\int_X^{\oplus} \bigcap_{s\geq t\ s\to\infty} \overline{\mathrm{Im}}[\bar{H}^1_{(2)}(\Sigma_{s,x})\to\bar{H}^1_{(2)}(\Sigma_{t,x})]\,d\mu(x) \simeq \int_X^{\oplus} \mathbf{HD}(\mathcal{G}_x)/\mathbb{C}\ d\mu(x) \qquad (13)$$

so that its $\mathcal{R}$-dimension does not depend on t, and Theorem 7.5 is proved by putting the equalities (11), (12) and (13) together. ∎

8 Some Background about Measured Equivalence Relations

In this section, we just recall briefly the definition of some notions appearing in the paper. The reader may consult [FM77] and [Gab00, Gab02] for more details and more references.

Countable standard equivalence relation. A *countable standard equivalence relation* on the standard Borel space (X,μ) is an equivalence relation $\mathcal{R}$ with countable classes that is a Borel subset of $X\times X$ for the product σ-algebra.

Preservation of the measure. The (countable standard) equivalence relation $\mathcal{R}$ is said to *preserve the measure* if for every partially defined isomorphism $\varphi : A\to B$ whose graph is contained in $\mathcal{R}$ ($\{(x,\varphi(x)) : x\in A\}\subset\mathcal{R}$), one has $\mu(A)=\mu(B)$, or equivalently iff the measures ν_1

and ν_2 on the set $\mathcal{R} \subset X \times X$ coincide, defined with respect to the projections on the first (resp. second) coordinate pr_1 (resp. pr_2) by $\nu_1(C) = \int_X \#(C \cap pr_1^{-1}(x))d\mu(x)$ and $\nu_2(C) = \int_X \#(C \cap pr_2^{-1}(y))d\mu(y)$. One denotes by $\nu = \nu_1 = \nu_2$ this common (usually infinite) measure on $\mathcal{R}$.

Essentially Free Action. A Borel action of H on a standard probability measure space (X, μ) is *essentially free* if the Borel subset of points $x \in X$ with non-trivial stabilizer ($\mathrm{Stab}_H(x) = \{h \in H : hx = x\} \neq \{id\}$) has μ-measure 0. The term "essentially" is frequently omitted.

Restrictions. Let (X, μ) be a standard Borel space with a probability measure μ and $\mathcal{R}$ a measure-preserving Borel equivalence relation. If Y is a Borel subset of X of non-zero measure, denote by $\mu_Y := \frac{\mu_{|Y}}{\mu(Y)}$ the normalized probability measure on Y. The *restriction* $\mathcal{R}_Y$ of $\mathcal{R}$ to Y is the μ_Y-measure-preserving Borel equivalence relation on Y defined by for every $x, y \in Y$, $x\mathcal{R}_Y y \Leftrightarrow x\mathcal{R}y$.

Saturation. A Borel subset $U \subset X$ is called $\mathcal{R}$-*saturated* if it is a union of $\mathcal{R}$-classes. The $\mathcal{R}$-saturation of a Borel set U is the smallest $\mathcal{R}$-saturated set containing it. It is the union of the $\mathcal{R}$-classes meeting U.

Finite index. A sub-equivalence relation $\mathcal{S} \subset \mathcal{R}$ has *finite index* in $\mathcal{R}$ if each $\mathcal{R}$-class decomposes into finitely many $\mathcal{S}$-classes. If this number is constant, it is called the index of $\mathcal{S}$ in $\mathcal{R}$ and is denoted by $[\mathcal{R} : \mathcal{S}]$.

Graphings. A *probability measure-preserving oriented graphing* on (X, μ) is an at most countable family $\Phi = (\varphi_i)_{i \in I}$ of partial measure-preserving isomorphisms $\varphi_i : A_i \to B_i$ between Borel subsets $A_i, B_i \subset X$.

A *probability measure-preserving unoriented graphing* Ψ on (X, μ) is a Borel subset of $X \times X \setminus \{(x, x) : x \in X\}$ that is symmetric under the flip $(x, y) \leftrightarrow (y, x)$ such that the smallest equivalence relation $\mathcal{R}_\Psi$ containing it has coutable classes and is measure-preserving. It provides a Borel choice of pairs of $\mathcal{R}_\Psi$-equivalent points ("neighbors"), and thus a graph structure on each equivalence class of $\mathcal{R}_\Psi$. When these graphs are (almost) all trees, the graphing is called a *treeing.*

$\mathcal{R}_\Psi$ is *generated* by Ψ.

Ψ is a graphing *over* $\mathcal{R}$ if it is contained in $\mathcal{R}$.

An oriented graphing defines clearly an unoriented one, by considering the graphs of the $\varphi_i, \varphi_i^{-1}$'s. The terms probability measure-preserving, oriented and unoriented are frequently omited.

The notion of *unoriented graphing* has been introduced by S. Adams in [Ada90] and *oriented graphing* by G. Levitt, together with the notion of *cost*, in [Lev95].

Cost. The *cost* of an unoriented graphing Ψ is the number $\mathrm{cost}(\Psi, \mu) := \frac{1}{2}\nu(\Psi)$, where ν is the witness measure on $\mathcal{R}_\Psi$ for $\mathcal{R}_\Psi$ to preserve the measure μ of X.

The cost of an oriented graphing $\Phi = (\varphi_i)_{i \in I}$, is the sum of the measures of the domains $\sum_{i \in I} \mu(A_i)$.

Except in the obvious cases (redundancy in Φ), the two notions coincide. In general the cost of an oriented graphing is greater than that of the associated unoriented one.

The *cost (*$\mathrm{cost}(\mathcal{R}, \mu)$*) of a p.m.p. countable equivalence relation* $\mathcal{R}$ is the infimum of the costs of the generating graphings. The *cost of a group* Γ is the infimum of $\mathrm{cost}(\mathcal{R}, \mu)$ over all equivalence

relations $\mathcal{R}$ defined by a p.m.p. free actions of Γ (see [Gab00]). A comparison has been established between the cost and the first L^2 Betti number: $\beta_1(\mathcal{R},\mu) \leq \mathrm{cost}(\mathcal{R},\mu) - 1$ [Gab02, Cor. 3.23]. Despite that equality is not known to be true in general for an $\mathcal{R}$ with only infinite classes, there is no (not yet ?) counterexample. However, when Ψ is a treeing, then $\beta_1(\mathcal{R}_\Psi,\mu) = \mathrm{cost}(\mathcal{R}_\Psi,\mu) - 1 = \mathrm{cost}(\Psi) - 1$ ([Gab02, Cor. 3.23] and [Gab00, Th. 1]).

References

[Ada90] S. Adams. Trees and amenable equivalence relations. *Ergodic Theory Dynamical Systems*, 10(1):1–14, 1990.

[Ati76] M. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, pages 43–72. *Astérisque, SMF*, No. 32–33. Soc. Math. France, Paris, 1976.

[BBCK00] M. Biskup, C. Borgs, J.T. Chayes, R. Kotecký. Gibbs states of graphical representations of the Potts model with external fields. *J. Math. Phys.*, 41(3): 1170–1210, 2000.

[BB99] E. Babson, I. Benjamini. Cut sets and normed cohomology with applications to percolation. *Proc. Amer. Math. Soc.*, 127(2):589–597, 1999.

[BK89] R. Burton, M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.

[BLPS99a] I. Benjamini, R. Lyons, Y. Peres, O. Schramm. Group-invariant percolation on graphs. *Geom. Funct. Anal.*, 9(1):29–66, 1999.

[BLPS99b] I. Benjamini, R. Lyons, Y. Peres, O. Schramm. Critical percolation on any nonamenable group has no infinite clusters. *Ann. Probab.*, 27(3):1347–1356, 1999.

[BLS99] I. Benjamini, R. Lyons, O. Schramm. Percolation perturbations in potential theory and random walks in *Random walks and discrete potential theory (Cortona, 1997)*, pages 56–84, Sympos. Math., XXXIX, Cambridge Univ. Press, 1999.

[BS96] I. Benjamini, O. Schramm. Percolation beyond $\mathbb{Z}^d$, many questions and a few answers. *Electron. Comm. Probab.*, 1:no. 8, 71–82 (electronic), 1996.

[BS01] I. Benjamini, O. Schramm. Percolation in the hyperbolic plane. *Jour. Amer. Math. Soc.*, 14: 487–507, 2001.

[BV97] M. Bekka, A. Valette. Group cohomology, harmonic functions and the first L^2-Betti number. *Potential Anal.*, 6(4):313–326, 1997.

[CG86] J. Cheeger, M. Gromov. L_2-cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.

[FK72] C.M. Fortuin, P.W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. *Physica*, 57, 536–564, 1972.

[FM77] J. Feldman, C. Moore. Ergodic equivalence relations, cohomology and von Neumann algebras. I and II. *Trans. Amer. Math. Soc.*, 234(2):289–324, 325–359, 1977.

[Gab00] D. GABORIAU. Coût des relations d'équivalence et des groupes. *Invent. Math.*, 139(1):41–98, 2000.

[Gab02] D. GABORIAU. Invariants L^2 de relations d'équivalence et de groupes. *Publ. math. Inst. Hautes Étud. Sci.*, 95(1):93–150, 2002.

[GN90] G. R. GRIMMETT, C. M. NEWMAN. Percolation in $\infty+1$ dimensions. In *Disorder in physical systems*, Oxford Sci. Publ., pages 167–190. Oxford Univ. Press, New York, 1990.

[HJL02a] O. HÄGGSTRÖM, J. JONASSON, R. LYONS. Coupling and Bernoullicity in random-cluster and Potts models. *Bernoulli*; 8(3): 275–294, 2002.

[HJL02b] O. HÄGGSTRÖM, J. JONASSON, R. LYONS. Explicit isoperimetric constants and phase transitions in the Random-Cluster Model. *Ann. of Proba.* 30 (1):443–473, 2002.

[HP99] O. HÄGGSTRÖM, Y. PERES. Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously. *Probab. Theory Related Fields*, 113(2):273–285, 1999.

[HPS99] O. Häggström, Y. Peres, R. H. Schonmann. Percolation on transitive graphs as a coalescent process: relentless merging followed by simultaneous uniqueness. In *Perplexing problems in probability*, volume 44 of *Progr. Probab.*, pages 69–90. Birkhäuser Boston, Boston, MA, 1999.

[Kec95] A. KECHRIS. Classical descriptive set theory, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.

[Lev95] G. LEVITT. On the cost of generating an equivalence relation. *Ergodic Theory Dynam. Systems*, 15(6):1173–1181, 1995.

[Lyo95] R. LYONS. Random walks and the growth of groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 320:1361–1366, 1995.

[LP05] R. LYONS, Y. PERES. Probability on Trees and Networks. Book in preparation. To appear. Cambridge University Press. http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html.

[LS99] R. LYONS, O. SCHRAMM. Indistinguishability of percolation clusters. *Ann. Probab.*, 27(4):1809–1836, 1999.

[Lyo00] R. LYONS. Phase transitions on nonamenable graphs. J. Math. Phys. 41, 1099–1126. Probabilistic techniques in equilibrium and nonequilibrium statistical physics, 2000.

[Mar03] F. MARTIN. Analyse harmonique et 1-cohomologie réduite des groupes localement compacts. Thèse de l'université de Neuchâtel, 2003.

[Per00] Y. PERES Percolation on nonamenable products at the uniqueness threshold. Ann. Inst. H. Poincaré Probab. Statist.36(5), 395–406, 2000.

[PP00] Y. PERES, R. PEMANTLE Nonamenable products are not treeable. Israel J. Math.118, 147–155, 2000.

[PSN00] I. PAK, T. SMIRNOVA-NAGNIBEDA. On non-uniqueness of percolation on nonamenable Cayley graphs. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(6):495–500, 2000.

[Sau03] R. SAUER. L^2-Betti Numbers of Discrete Measured Groupoids. Preprint arXiv math.AT/0312411.

[Soa93] P. M. SOARDI. Rough isometries and Dirichlet finite harmonic functions on graphs. *Proc. Amer. Math. Soc.*, 119(4):1239–1248, 1993.

D. G.: UMPA, UMR CNRS 5669, ENS-LYON, 69364 LYON CEDEX 7, FRANCE
gaboriau@umpa.ens-lyon.fr